Fourier Finite Element Methods for an Elliptic State Constrained Optimal Control Problem on Axisymmetric Domains

Andre Mas

Advisor: Minah Oh (ohmx@jmu.edu)

andremas45@gmail.com

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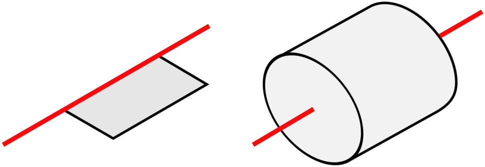
Acknowledgements

The material in this presentation arose from an open ended class project in math-449 (Numerical Differential Equations), taught by Dr. Minah Oh (ohmx@jmu.edu) during the Fall semester of 2022 at James Madison University. This is a continuation of the work of several other JMU students in (G. Henderson, M. Oh, J. Spangler).

Thank you to Dr. Oh for her guidance and patience over the course of this school year.

A Motivating Scenario

An **axisymmetric** 3D domain $\breve{\Omega}$ is obtained by rotating a 2D domain Ω.



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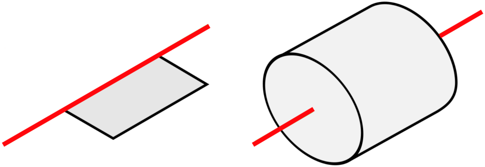

Goal: Perform a dimension reduction to an axisymmetric elliptic state constrained optimal control problem by using cyclindrical coordinates and a Fourier series decomposition. Finite element methods may then be used to efficiently solve the resulting 2D problems, which are the Fourier modes of the original 3D problem. New Fourier finite element spaces will be used for non-zero Fourier modes.

High Level Overview

Problem defined on the 3d domain $\breve{\Omega}$

$\downarrow$

Sequence of problems on the 2d domain Ω

$\downarrow$

2d problem discretization

$\downarrow$

Numerical Results

3d ($\breve{\Omega}$) Problem Formulation

$$(y^\star, u^\star) = \operatorname*{argmin}_{(y,u)\in\mathbb{K}} \left[\frac{1}{2}\|y - y_d\|^2_{L^2(\breve{\Omega})} + \frac{\beta}{2}\|u\|^2_{L^2(\breve{\Omega})}\right] \tag{1}$$

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where (y, u) belong to $\mathbb{K} \subset H^1(\breve{\Omega}) \times L^2(\breve{\Omega})$ if and only if

$$\int_{\breve{\Omega}} \nabla y \cdot \nabla w + \int_{\breve{\Omega}} yw = \int_{\breve{\Omega}} uw \quad \forall\, w \in H^1(\breve{\Omega}) \tag{2}$$

and $y \leq \psi$ a.e in $\breve{\Omega}$. The functions y_d and ψ, as well as the constant β are given.

Partial Fourier Series Decomposition

Let $U(r, z, \theta) \in L^2(\breve{\Omega})$. Then U may be expanded (PFSD) as:

$$U = U_0 + \sum_{k=1}^{\infty} U_k \cos k\theta + \sum_{k=1}^{\infty} U_{-k} \sin k\theta$$

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We denote by $\breve{L}^2(\breve{\Omega}) \subseteq L^2(\breve{\Omega})$ the set of all axisymmetric functions defined on $\breve{\Omega}$. It may be shown that $\breve{L}^2(\breve{\Omega}) \cong L^2_r(\Omega)$.

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Takeaway: Each Fourier mode U_k is defined over Ω, not $\breve{\Omega}$. Solving multiple 2d problems is “easier” than solving a 3d problem.

The Gradient Operator for the k-th Fourier Mode

$$\mathbf{grad}_{rz}^{k} U_k = \begin{bmatrix} \partial_r U_k \\ -\frac{k}{r} U_k \\ \partial_z U_k \end{bmatrix}$$

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This is related to the gradient operator in cylindrical coordinates. We now define the space:

$$H_r(\mathbf{grad}_k\ ,\ \Omega) = \left\{ u \in L_r^2(\Omega) : \mathbf{grad}_{rz}^{k} u \in L_r^2(\Omega) \times L_r^2(\Omega) \times L_r^2(\Omega) \right\}$$

A Sequence of 2d (Ω) Problems

$$(y_k{}^\star, u_k{}^\star) = \underset{(y_k, u_k) \in \mathbb{K}^k}{\operatorname{argmin}} \left[\frac{1}{2} \| y_k - y_d^k \|^2_{L^2_r(\Omega)} + \frac{\beta}{2} \| u_k \|^2_{L^2_r(\Omega)} \right] \tag{3}$$

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where (y_k, u_k) belong to $\mathbb{K}^k \subset H_r(\mathbf{grad}_k\ ,\ \Omega) \times L^2_r(\Omega)$ if and only if:

$$\int_\Omega (\mathbf{grad}^k_{rz} y_k \cdot \mathbf{grad}^k_{rz} w_k) r dr dz + \int_\Omega (y_k w_k) r dr dz = \int_\Omega (u_k w_k) r dr dz \quad \forall\, w_k \in H_r(\mathbf{grad}_k\ ,\ \Omega)$$

and $y_k \leq \psi_k$ a.e in Ω.

Fourier Mode $k = 0$

Numerical examples for the Fourier mode $k = 0$ case were computed using a standard P1 FEM scheme in (G. Henderson, M. Oh, J. Spangler).

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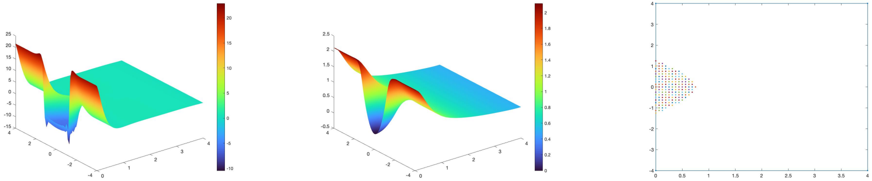
Example 1: $\beta = .002$, $y_d^0(r, z) = 1/\sqrt{r}$, $\psi_0(r, z) = r^2 + z^2$, $\Omega = [0, 4] \times [-4, 4]$.

u_h (control), y_h (state), contact set



Fourier Mode $k = 0$ Error Chart

h	$\|\|y_{h+1} - y_h\|\|_{L^2_r(\Omega)}$	rate	$\|\|y_{h+1} - y_h\|\|_{H^1_r(\Omega)}$
1	5.53e+00		1.57e+00
2	1.30e+00	2.08	2.46e+00
3	1.26e+00	0.04	3.07e+00
4	3.27e-01	1.94	1.44e+00
5	1.64e-01	0.99	1.01e+00
6	4.36e-02	1.90	5.33e-01
7	1.17e-02	1.89	2.73e-01

For further numerical examples (Fourier Mode = 0), see: (G. Henderson, M. Oh, J. Spangler).

A New Discretization for $k \neq 0$

Let $\mathscr{T}_h$ be a triangulation of Ω, and $V_h \subset H^1(\Omega)$ be the P^1 finite element space associated with $\mathscr{T}_h$.

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$$V_h^k = \{v_h \in H_r(\mathbf{grad}^k, \Omega) : v_h|_\triangle \in A \text{ for all } \triangle \in \mathscr{T}_h\} \tag{4}$$

where $A = \{ar^2 + brz + cr : a, b, c \in \mathbb{R}\}$.

2d Discrete Problem Formulation

$$(y_h^{k\star}, u_h^{k\star}) = \underset{(y_h^k, u_h^k)\in W_h}{\operatorname{argmin}} \left[\frac{1}{2}\|y_h^k - y_d^k\|^2_{L_r^2(\Omega)} + \frac{\beta}{2}\|u_h^k\|^2_{L_r^2(\Omega)}\right] \tag{5}$$

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where (y_h^k, u_h^k) belong to W_h if and only if:

$$\int_\Omega (\mathbf{grad}_{rz}^k y_h^k \cdot \mathbf{grad}_{rz}^k w_k + y_h^k w_k) rdrdz = \int_\Omega (u_h^k w_k) rdrdz \quad \forall\, w_k \in H_r(\mathbf{grad}_k\,,\ \Omega)$$

and $y_h^k \leq \psi_h^k$ at each node in $\mathscr{T}_h$.

A Numerical Example for $k \neq 0$

Let $\beta = 1$ and $\Omega = [0, 4] \times [-4, 4]$, with the following functions given:

$$y_d(r, z, \theta) = r^2 + z^2 + \theta^2 - 1 \qquad \psi(r, z, \theta) = 2zr\sin(\theta)$$

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Let $\beta = 1$ and $\Omega = [0,4] \times [-4,4]$, with the following functions given:

$$y_d(r,z,\theta) = r^2 + z^2 + \theta^2 - 1 \qquad \psi(r,z,\theta) = 2zr\sin(\theta)$$

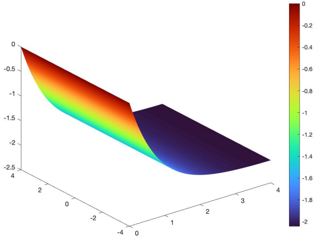
From these given functions, their Fourier modes are as follows:

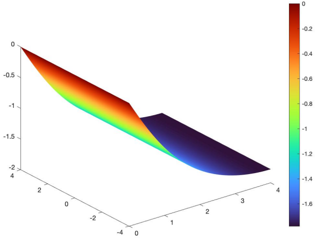
$$y_d^k(r,z) = \begin{cases} \frac{4\cos(k\pi)}{k^2} & k > 0 \\ \frac{2\pi(r^2+z^2-1)+(2\pi^3/3)}{\pi} & k = 0 \\ 0 & k < 0 \end{cases}$$

$$\psi_k(r,z) = \begin{cases} 0 & k > 0 \\ 0 & k = 0 \\ -2rz & k = -1 \\ \frac{-4rz\sin(k\pi)}{(k^2-1)\pi} & k < -1 \end{cases}$$

Fourier Mode $k = 1$

u_h (left), y_h (right)



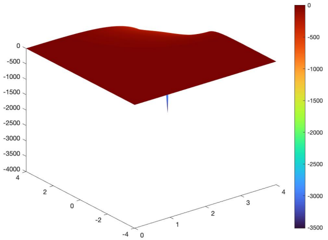


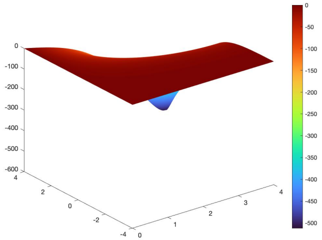
Fourier Mode $k = 1$ Error Chart

h	$\|\|y_{h+1} - y_h\|\|_{L^2_r(\Omega)}$	rate
1	3.79e+00	
2	9.09e-03	8.70
3	9.61e-03	-0.08
4	5.17e-04	4.21
5	1.34e-04	1.94
6	1.47e-05	3.18
7	2.01e-06	2.87

Fourier Mode $k = -1$

u_h (left), y_h (right)





Fourier Mode $k = -1$ Error Chart

h	$\|\|y_{h+1} - y_h\|\|_{L^2_r(\Omega)}$	rate
1	4.36e+02	
2	7.26e+01	2.58
3	6.07e-00	3.58
4	8.90e-01	2.76
5	1.70e-01	2.38
6	2.47e-02	2.78
7	1.69e-03	3.86

References

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