“Dangers” and surprises *of secret gift-giving games*

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April 29, 2017
MAA Sectional Meeting
Frostburg State University

Abstract:

Permutations and their cycles play an important role
is certain ice-breaker games and seasonal gift swaps. Often,
mathematical results surprise the participants. I look at a few
examples of such games where I've seen fellow participants surprised
by the results. Should we have been surprised?

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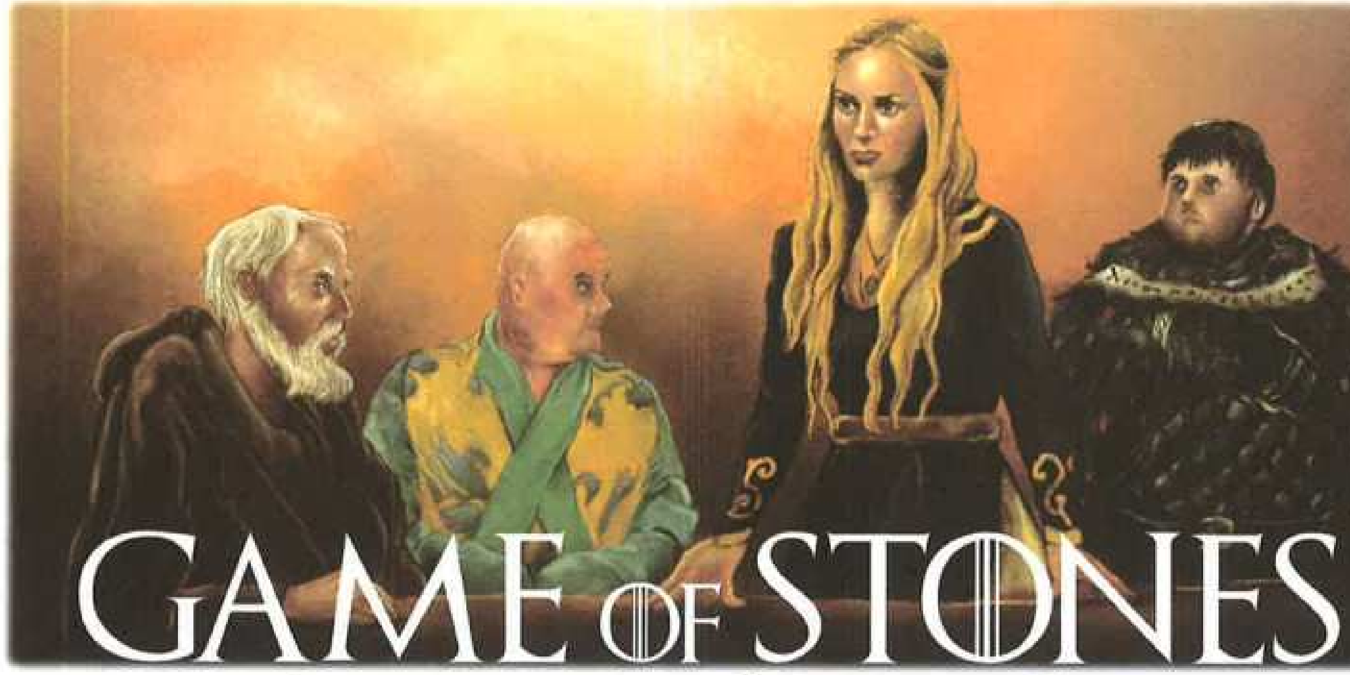

Illustrations by Michael Philip

A Tour of Knights

Dan Kalman and Dan Pritikin

Math Horizons article April 2017

Games

- Game of Stones
  - HIGHLY DANGEROUS
  - Expect to die
- Real world games
  - Social Danger Only
  - We can mitigate the dangers

“Game” Sources:

- Having students grade each others' quizzes
- Ice-breaker games where everyone has to “introduce” another random participant
- Secret gift-giving in which the same pairings

  are kept for several mysterious cycles

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- Ice-breaker games where everyone has to “introduce” another random participant
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  are kept for several mysterious cycles

Mathematically the same setups,
but played differently.
So the social dangers are different
in each game.

Grading Each Other's Papers

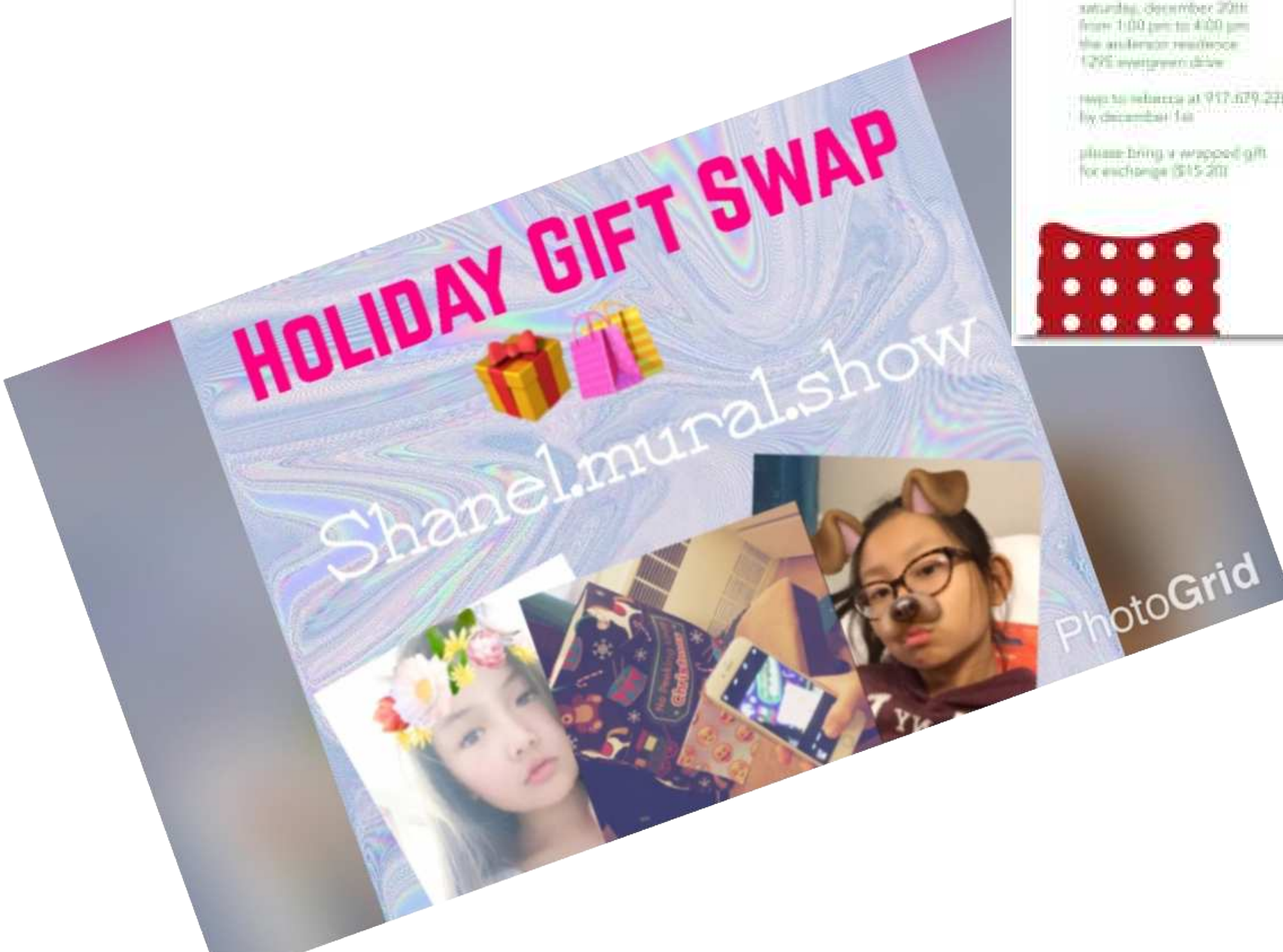
Is it actually still legal?

- 2001 Falvo v. Owasso: This is illegal because of FERPA. (Unless you anonymize it.)
- 2002 Supreme Court: Nah. It's all good!

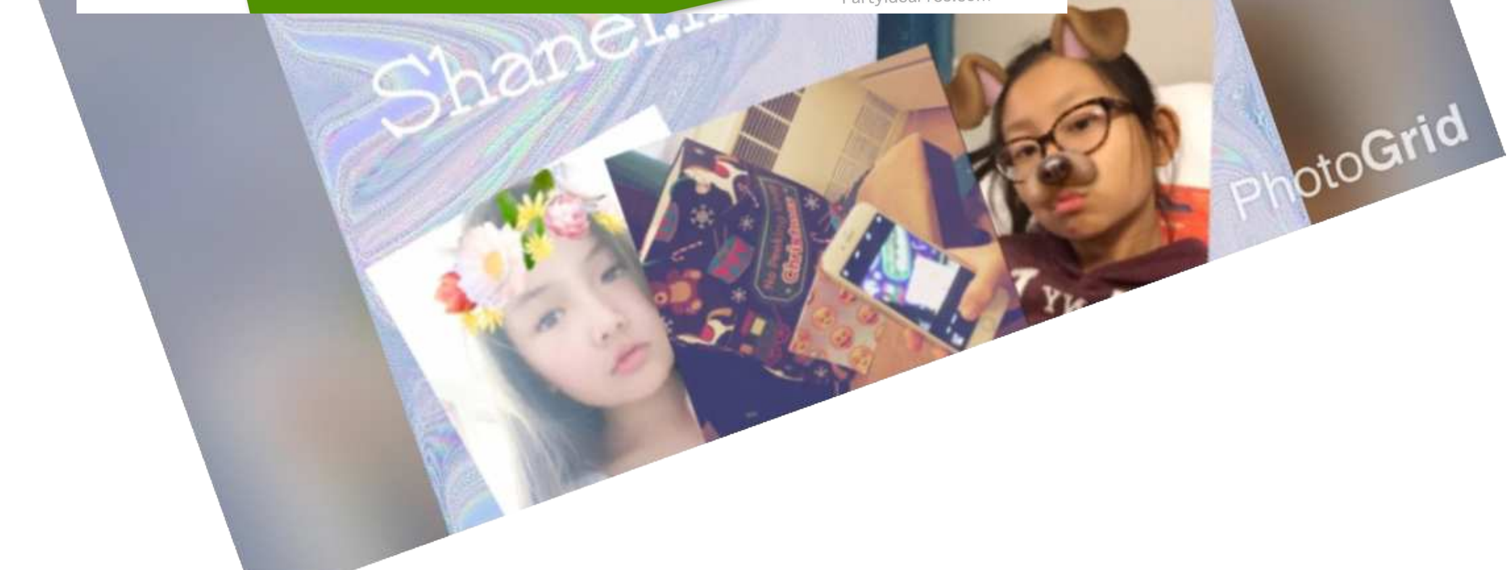
Mothers' Ice-Breaker

“Secret Sister”

“Secret Sister”

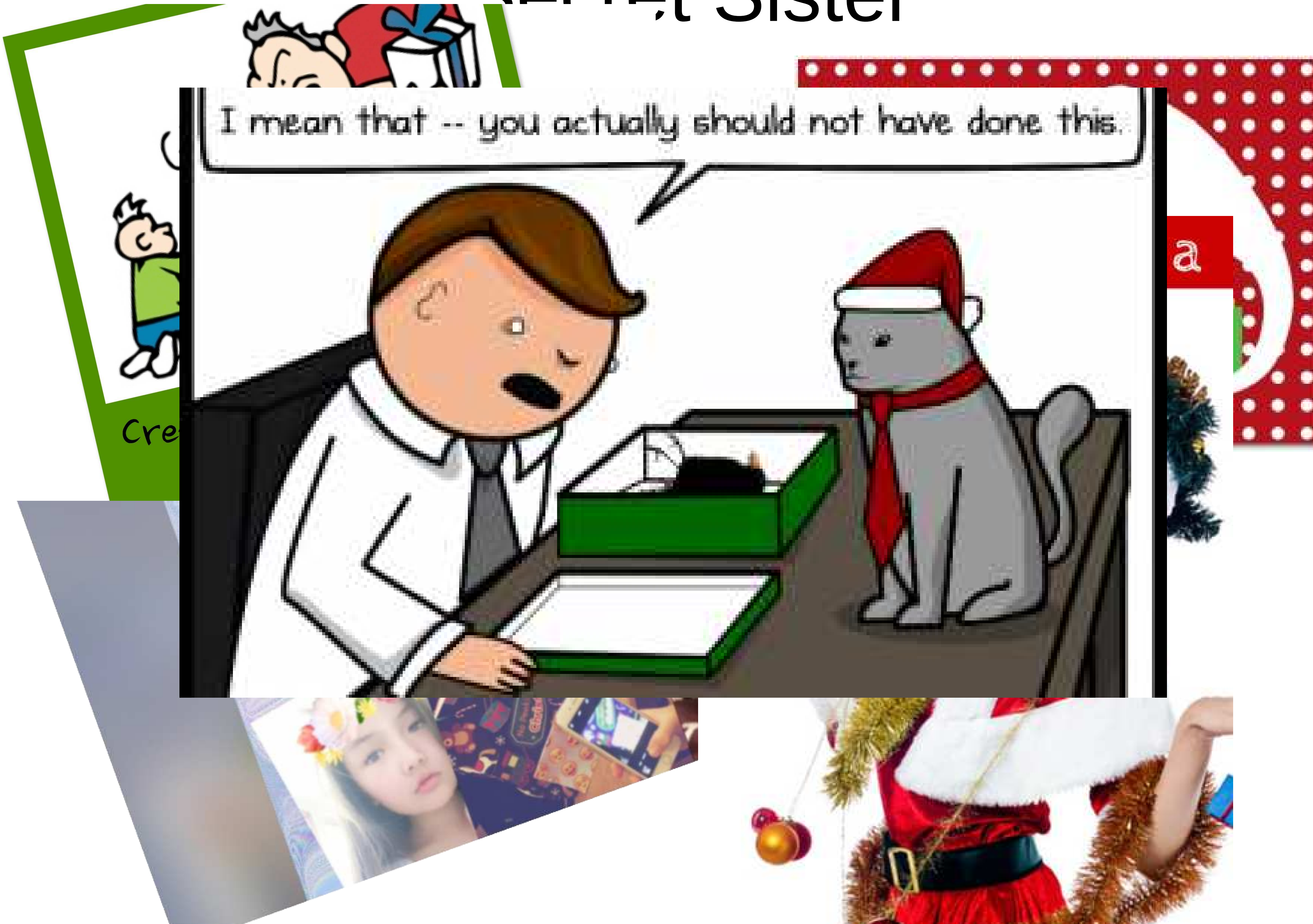


“Secret Sister”

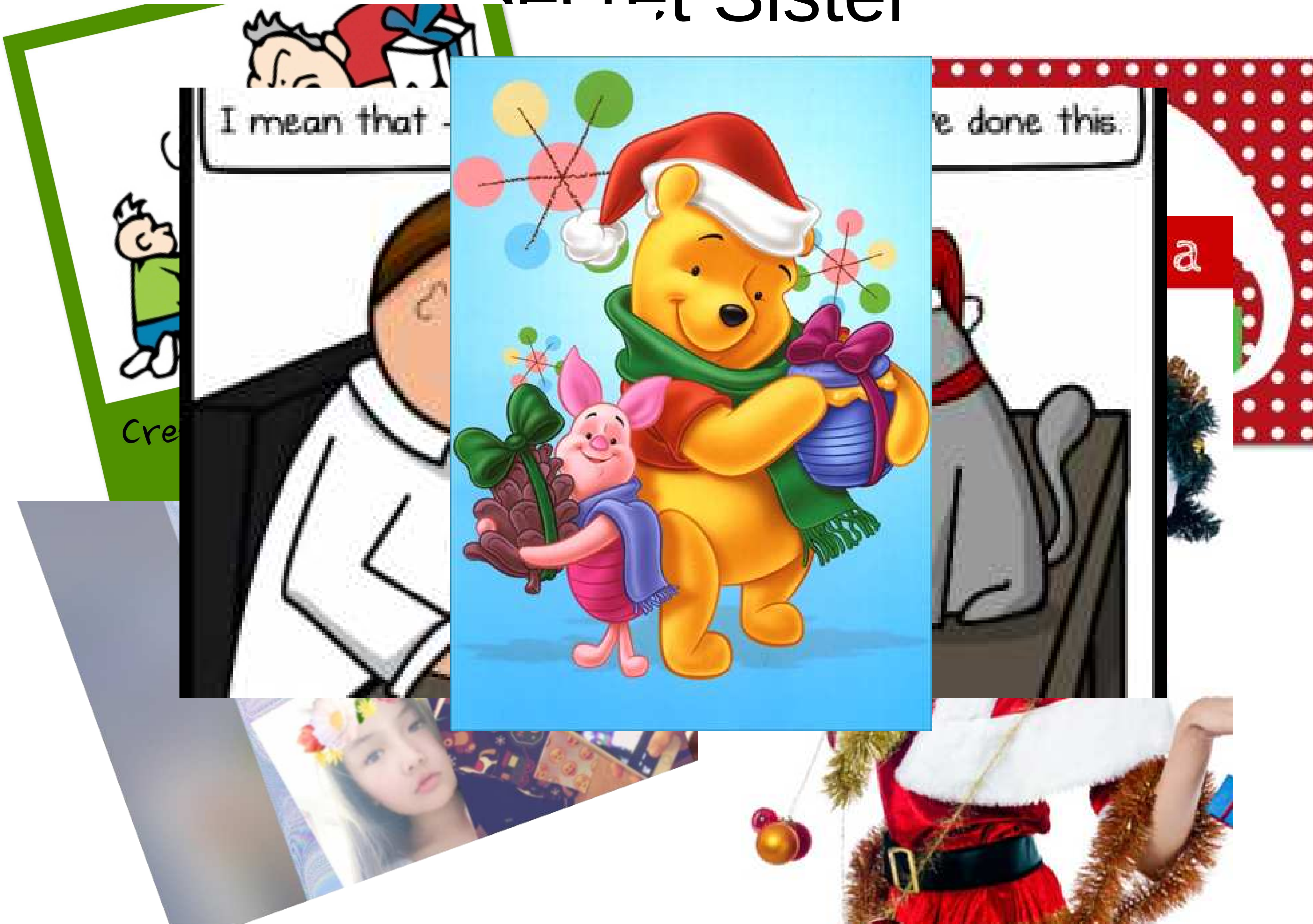


“Secret Sister”

“Secret Sister”



“Secret Sister”



First “Game”: Grading Papers

- If there are n students, *your* chances of getting your own paper: $1/n$
- Expected number of self-graders = 1.

But what is the distribution like?

- Two Student Case:
  - 50% chance of no self-graders
  - 50% chance of 2 self-graders

But what is the distribution like?

- Three Student Case:
  - abc 3
  - acb 1
  - bac 1
  - bca 0
  - cab 0
  - cba 1
- 17% chance of 3 self-graders
- 50% chance of 1 self-grader
- 33% chance of 0 self-graders

Think of it this way...

- Everyone could keep their own paper.
- There are three ways two students can swap papers while one person keeps their paper.
- Otherwise, they sit in a triangle and either pass their papers to the right or to the left (two ways)
- Total: 3!=6 ways to arrange the three papers

Four Student Case:

abcd 4	**abdc 2**	**acbd 2**	**acdb 1**	**adcb 2**	**adbc 1**
bacd 2	**badc 0**	**bcad 1**	**bcda 0**	**bdac 0**	**bdca 1**
cabd 1	**cadb 0**	**cbad 2**	**cbda 1**	**cdab 0**	**cdba 0**
dabc 0	**dacb 1**	**dbac 1**	**dbca 2**	**dcab 0**	**dcba 0**

9/24 no self-grading (How could we count this... hmmm?)
8/24 one self grader (Choose 1 of 4, then 2 ways for 3 others)
6/24 two self graders
0/24 three self graders (why is this zero?)
1/24 four self graders (why is this just one case?)

Do we care about self-graders?

- If a child gets their own paper, usually they won't be able to resist blowing their own cover.
- In most other games, self-matching is not fun so it is resolved by formal or informal methods.
- We will keep looking at all permutations but focus on the *derangements*.

A ***derangement*** is a permutation in which nothing gets to stay in the same place.

How can we count this?

- First how **not** to do this:
  - We have students ABCD
  - Give student A a paper to grade. 3 ways
  - Give student B a paper to grade....

How can we count this?

- First how **not** to do this:
  - We have students ABCD
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  - Give student B a paper to grade....
    - If A has B's paper (1/3 chance) then B has 3 options
    - Otherwise B has 2 options (2/3 chance)

How can we count this?

- First how **<u>not</u>** to do this:
  - We have students ABCD
  - Give student A a paper to grade. 3 ways
  - Give student B a paper to grade....
    - If A has B's paper (1/3 chance) then B has 3 options
    - Otherwise B has 2 options (2/3 chance)

YIKES!!! NO!!!

Is this ok?

- Line up the students randomly.
- Everyone passes their paper to the right.
  - The person on the end brings their paper to the other side of the line.
- So, really it's a circle.
  - So we have $n! / n = (n-1)!$ ways to do this

But Wait!

- This means that for 4 students,
  we should have 3!=6 derangements
- But I thought we counted 9 derangements!

Four Student Case:

abcd 4	**abdc 2**	**acbd 2**	**acdb 1**	**adcb 2**	**adbc 1**
bacd 2	**badc 0**	**bcad 1**	**bcda 0**	**bdac 0**	**bdca 1**
cabd 1	**cadb 0**	**cbad 2**	**cbda 1**	**cdab 0**	**cdba 0**
dabc 0	**dacb 1**	**dbac 1**	**dbca 2**	**dcab 0**	**dcba 0**

9/24 no self-grading (How could we count this... hmmm?)
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But Wait!

- This means that for 4 students,
  we should have $3!=6$ derangements
- But I thought we counted 9 derangements!
- Where do the other 3 derangements come from!?

Some Talk Objectives

- Realize that this is a common logical error!
  - (And it's kindof amusing at times!)
- Every circle of items makes a derangement
- But not every derangement is a single cycle!

The other derangements of 4 items

- Of the four items, select two to swap
- Swap the other two as well
- How many ways to do this?
  - AB CD
  - AC BD
  - AD BC
- If you want to swap two groups of two, you can calculate 4 choose 2, but that overcounts by a factor of two, because you have *two* groups of two.

There are formulas for all of this:

Theorem: Consider any partition P of n, with distinct terms n_j and frequencies f_j. Let $E_P = \{\pi \in S_n \mid$ the factor length partition for π is P. $\}$ Then

$$|E_P| = \frac{n!}{n_1^{f_1} n_2^{f_2} \cdots n_r^{f_2}} \frac{1}{f_1! f_2! \cdots f_r!} .$$

There are formulas for all of this:

$$n = \sum_{j=1}^{r} f_j n_j$$

Theorem: Consider any partition P of n, with distinct terms n_j and frequencies f_j. Let $E_P = \{\pi \in S_n \mid \text{the factor length partition for } \pi \text{ is } P.\}$ Then

$$|E_P| = \frac{n!}{n_1^{f_1} n_2^{f_2} \cdots n_r^{f_2}} \frac{1}{f_1! f_2! \cdots f_r!} .$$

There are formulas for all of this:

$$n = \sum^{r} f_j n_j$$

$$\binom{n}{n_1, n_1, \cdots, n_1, n_2, n_2, \cdots, n_2, \cdots, n_r, \cdots, n_r} = \frac{n!}{(n_1!)^{f_1}(n_2!)^{f_2} \cdots (n_r!)^{f_r}}.$$

… and frequencies f_j. Let

Theorem: Consider any …

$E_P = \{\pi \in S_n \mid$ the factor length par…

$$|E_P| = \frac{n!}{n_1^{f_1} n_2^{f_2} \cdots n_r^{f_r} \, f_1! \, f_2! \cdots}$$

There are formulas for all of this:

$$n = \sum^{r} f_j n_j$$

$$\binom{n}{n_1, n_1, \cdots, n_1, n_2, n_2, \cdots, n_2, \cdots, n_r, \cdots, n_r} = \frac{n!}{(n_1!)^{f_1}(n_2!)^{f_2} \cdots (n_r!)^{f_r}}.$$

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Theorem: Consider any ...
$E_P = \{\pi \in S_n \mid$ the factor length pa...

$$|E_P| = \frac{n!}{n_1^{f_1} n_2^{f_2} \cdots n_r^{f_2}} \cdots f_1!\, f_2! \cdots$$

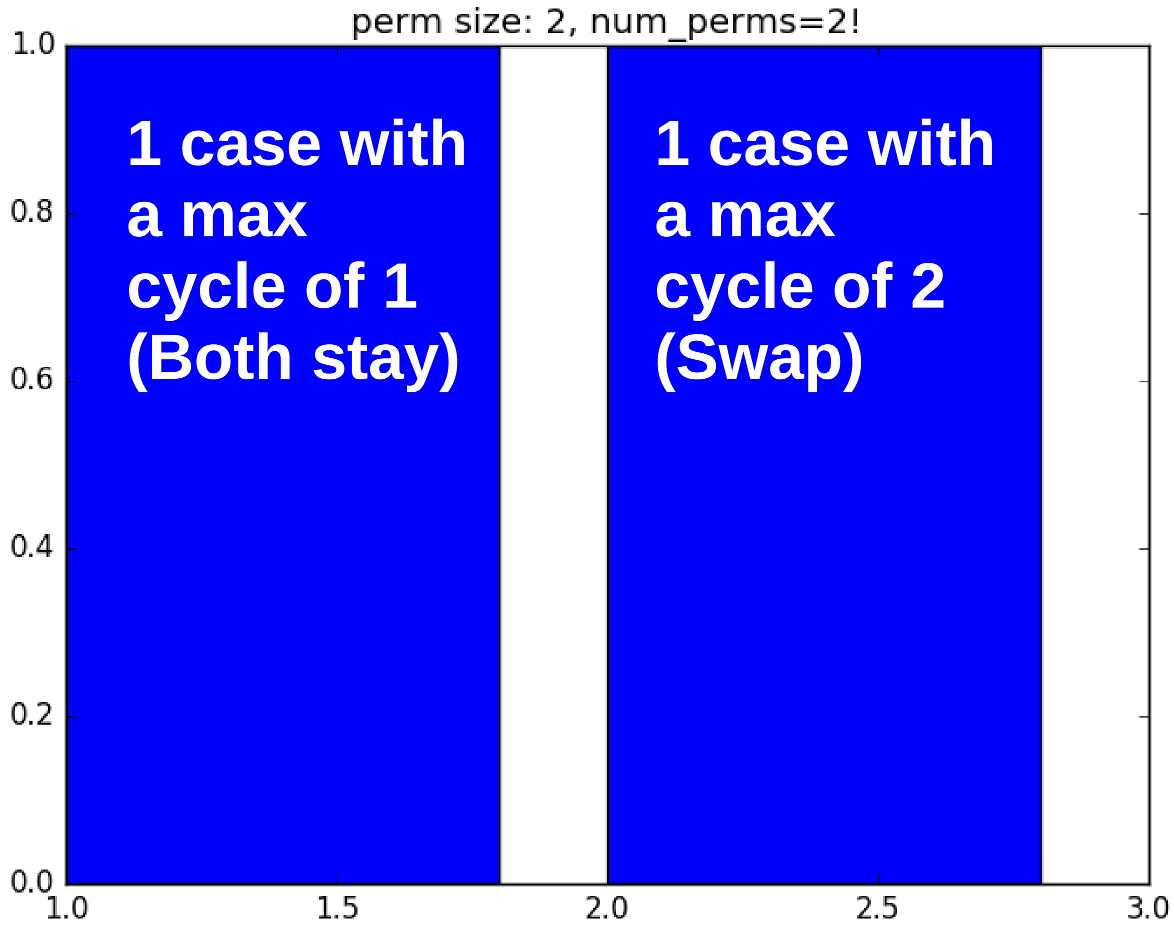
Thus, $|E_P| = \frac{n!}{n_1^{f_1} n_2^{f_2} \cdots n_r^{f_2}} \frac{1}{f_1!f_2!\cdots f_r!}$ completing the proof. ■

I'm sticking to numerics for this talk!

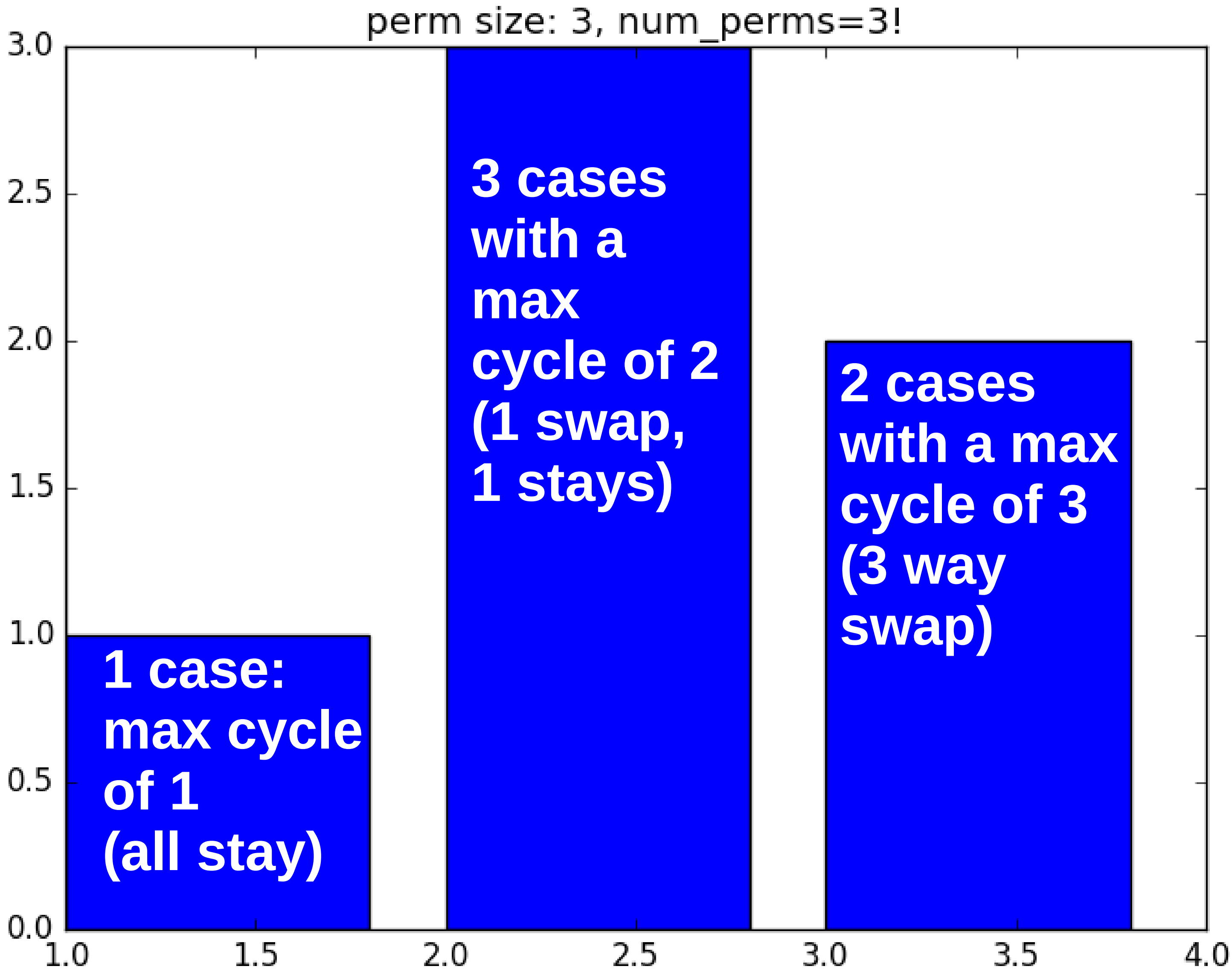
For all permutations of *n* items:

- We can look at permutations by their maximum cycle length.
- What do these distributions look like?

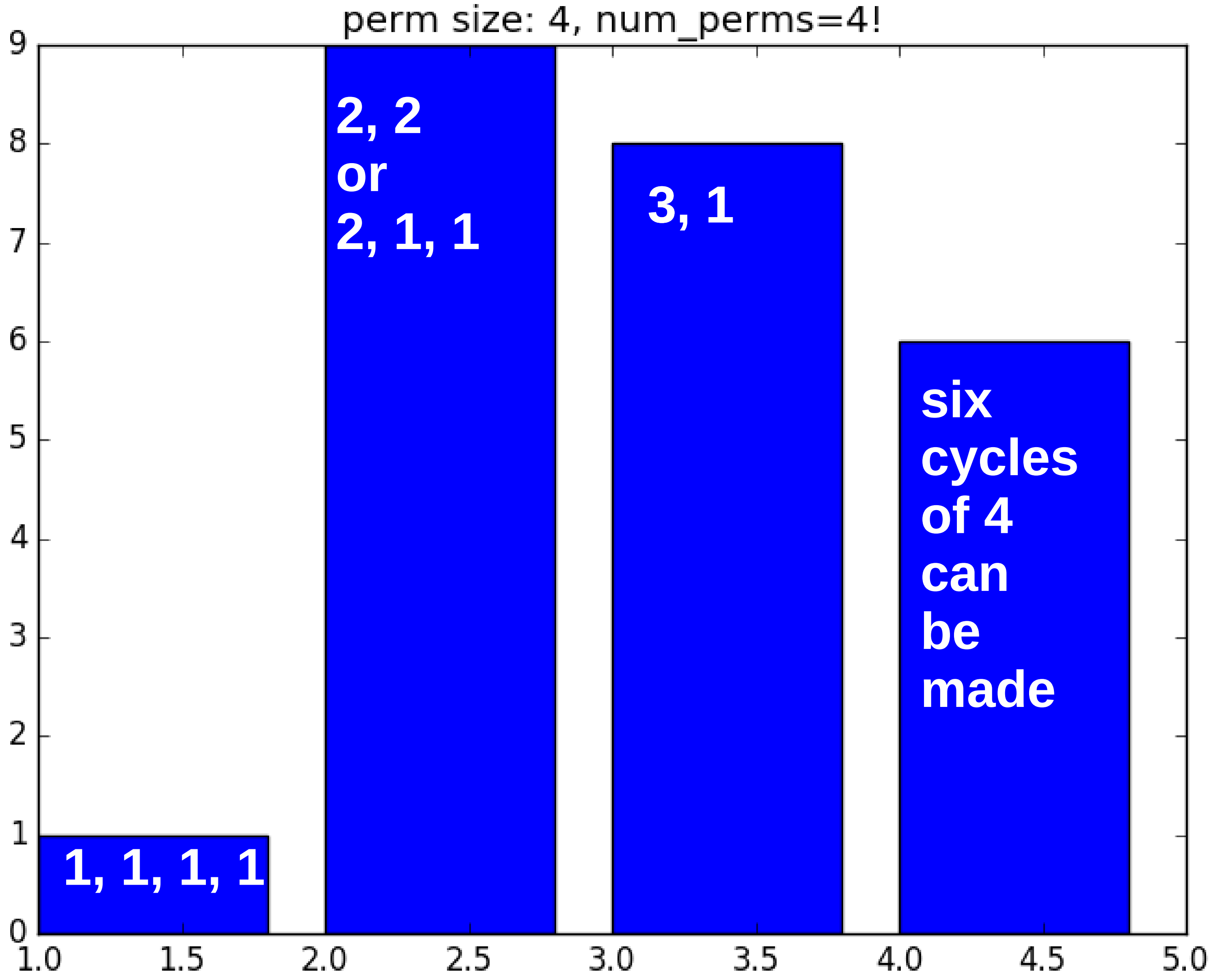
n=2 case


perm size: 2, num_perms=2!
1.0
0.8
0.6
0.4
0.2
0.0
1.0
1.5
2.0
2.5
3.0
1 case with a max cycle of 1 (Both stay)
1 case with a max cycle of 2 (Swap)

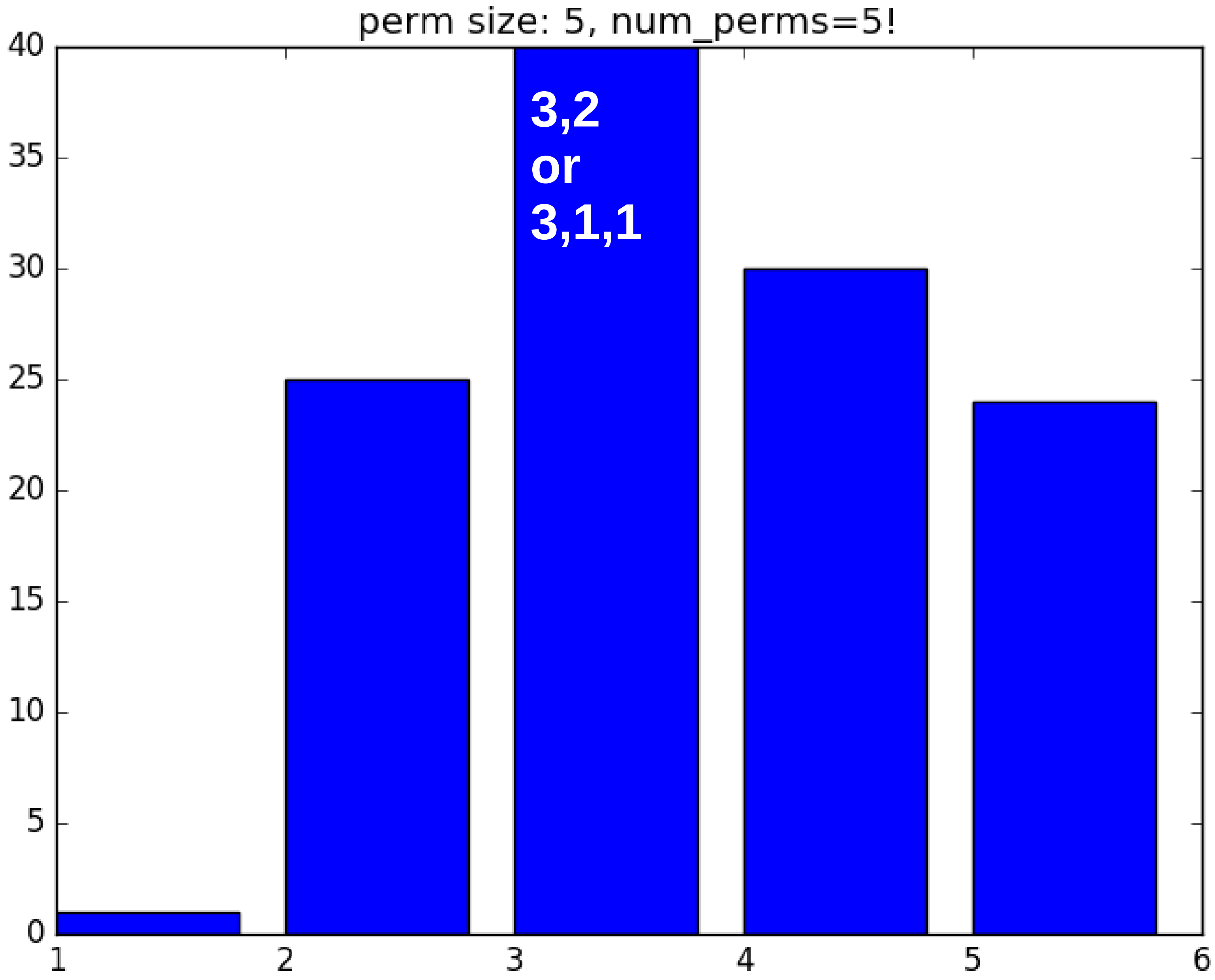
n=3 case


perm size: 3, num_perms=3!
3.0
2.5
2.0
1.5
1.0
0.5
0.0
1.0
1.5
2.0
2.5
3.0
3.5
4.0
1 case: max cycle of 1 (all stay)
3 cases with a max cycle of 2 (1 swap, 1 stays)
2 cases with a max cycle of 3 (3 way swap)

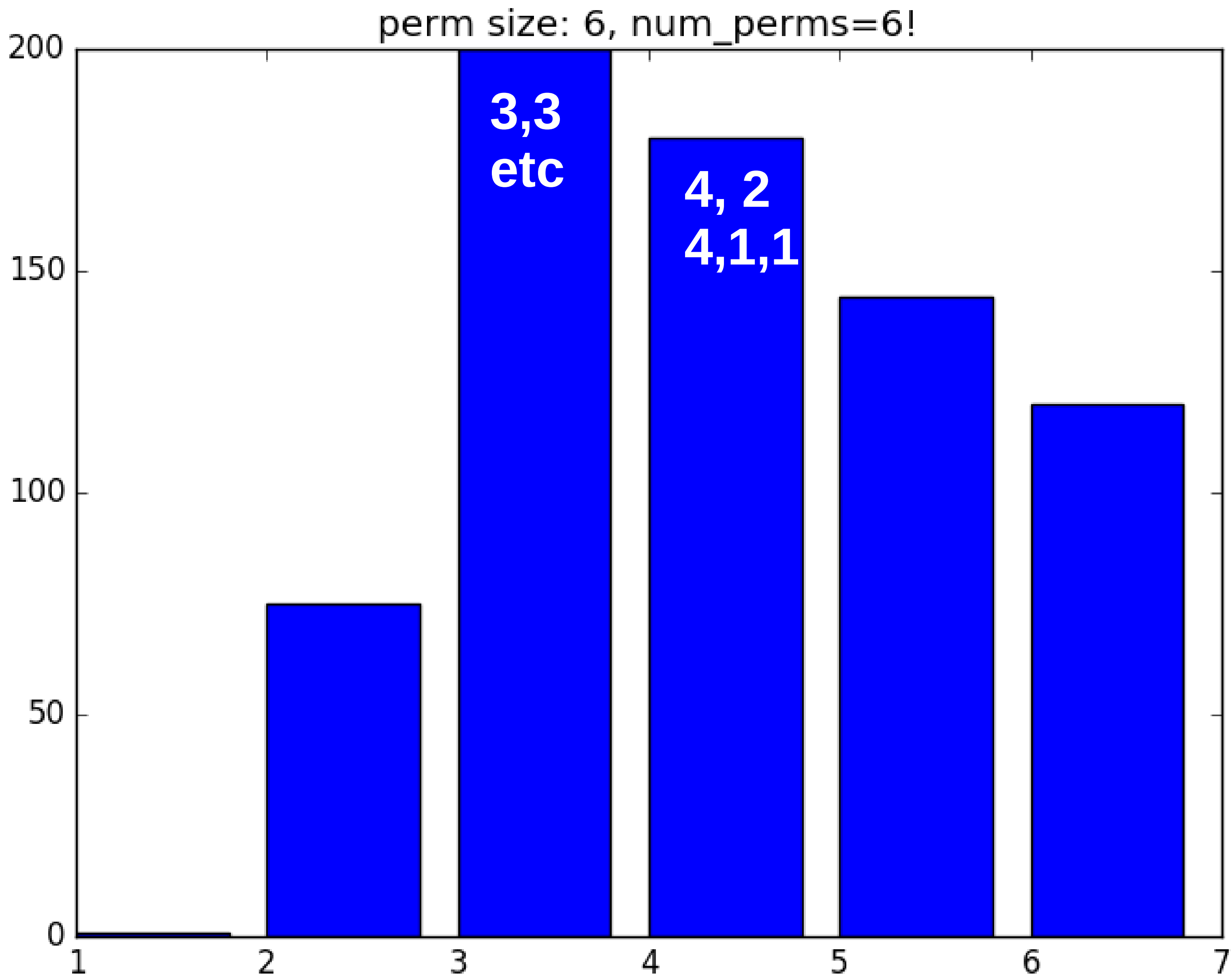
n=4 case


perm size: 4, num_perms=4!
9
8
7
6
5
4
3
2
1
0
1.0
1.5
2.0
2.5
3.0
3.5
4.0
4.5
5.0
1, 1, 1, 1
2, 2
or
2, 1, 1
3, 1
six
cycles
of 4
can
be
made

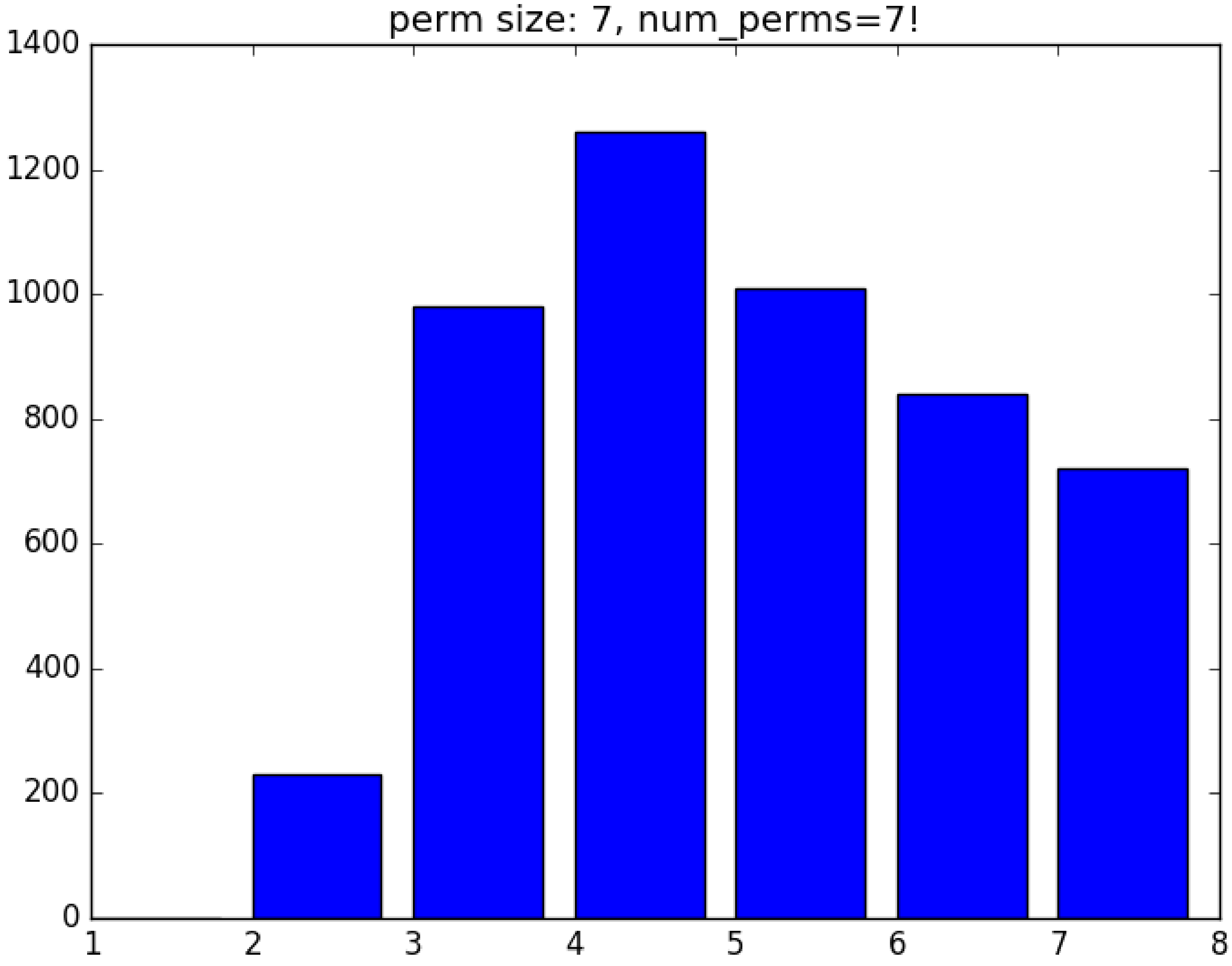
n=5 case



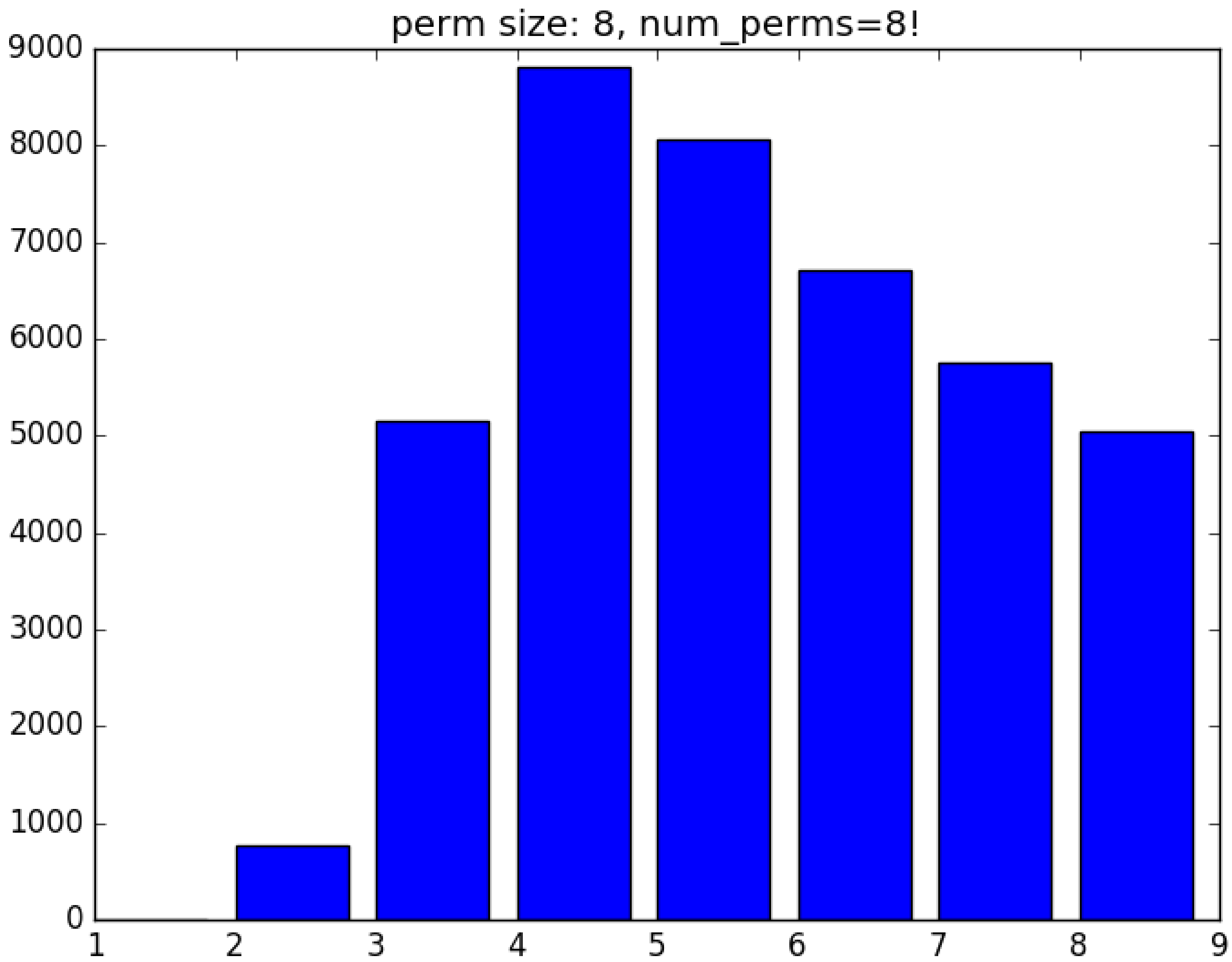
n=6 case



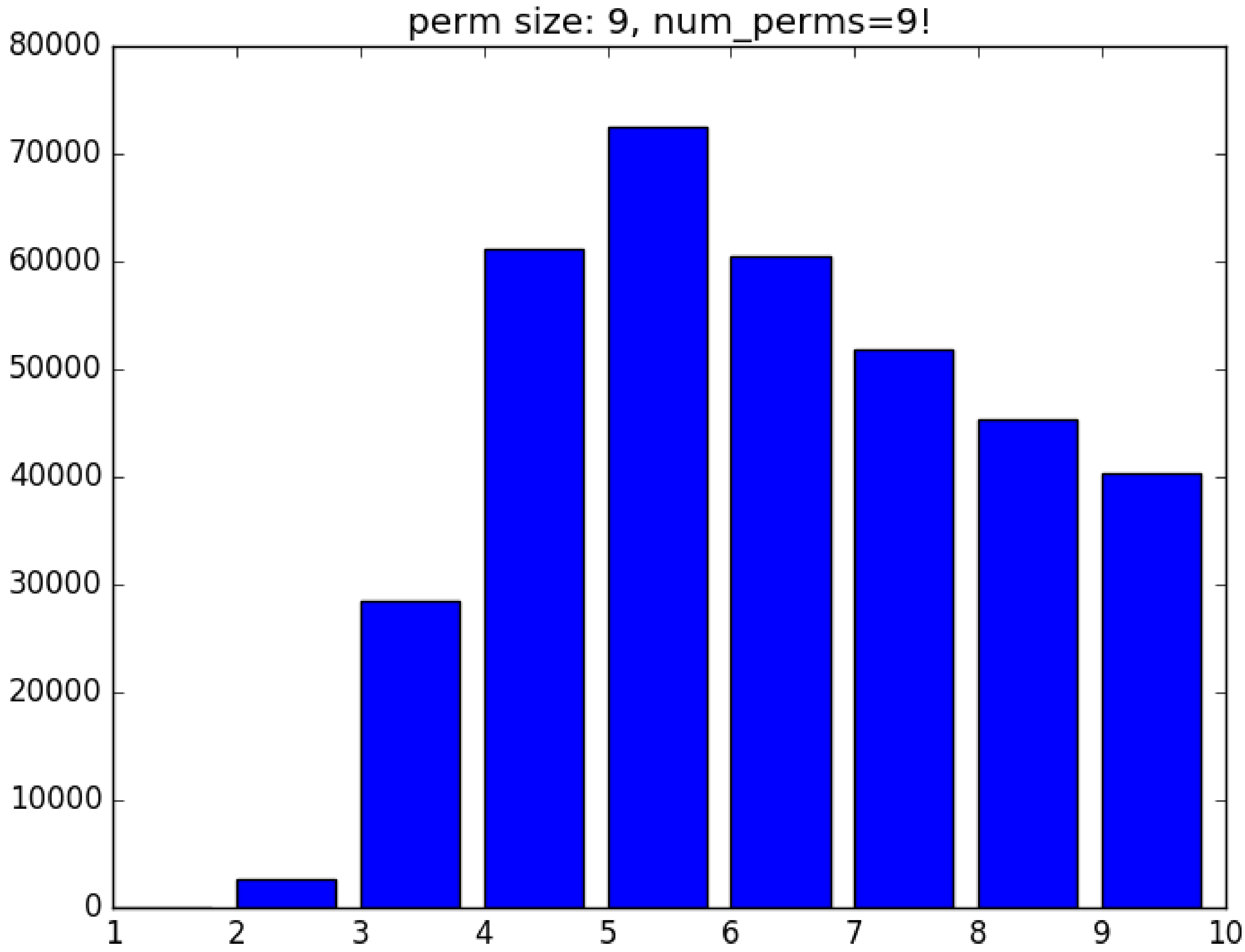
n=7 case


perm size: 7, num_perms=7!
1400
1200
1000
800
600
400
200
0
1
2
3
4
5
6
7
8

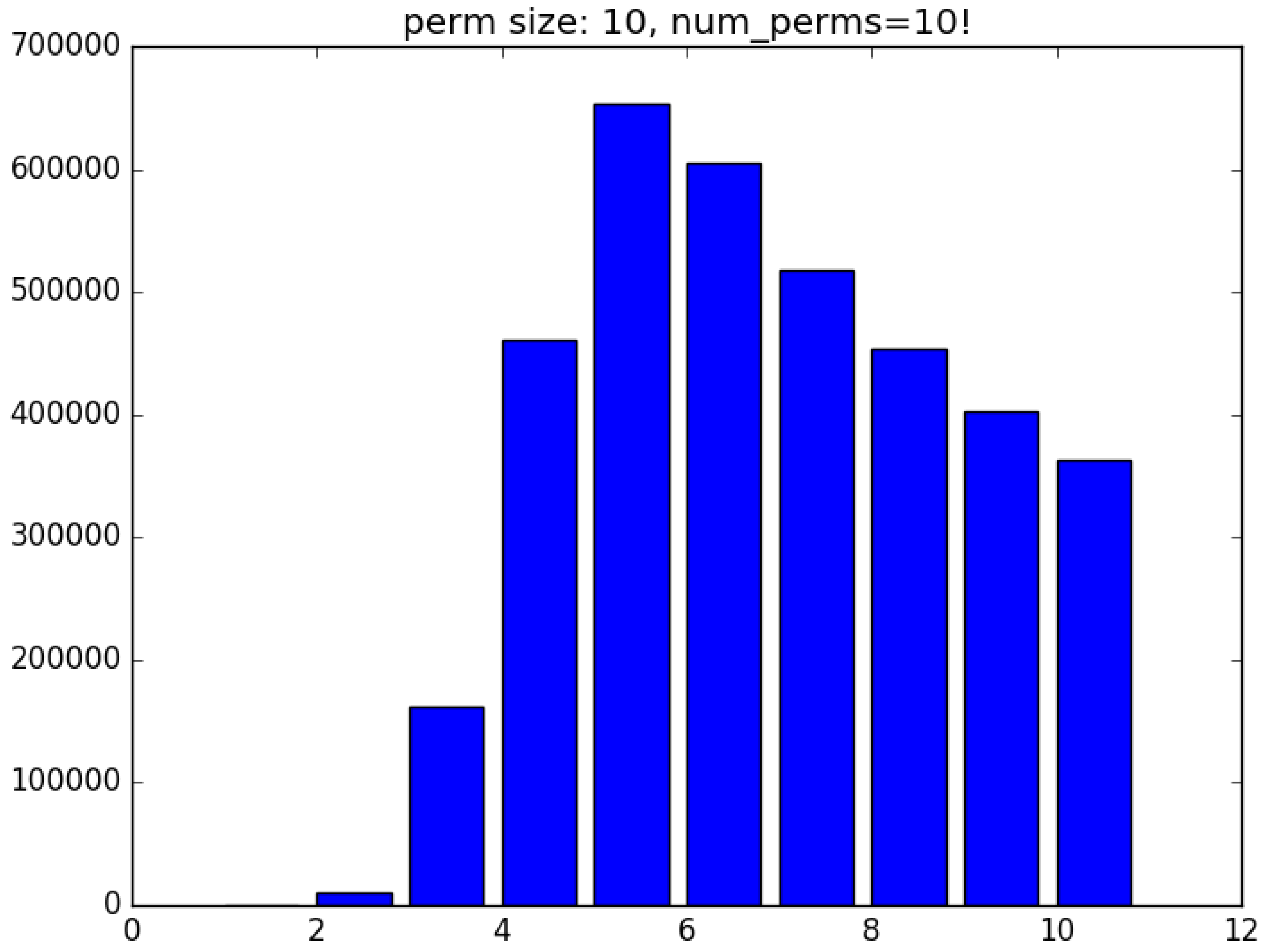
n=8 case

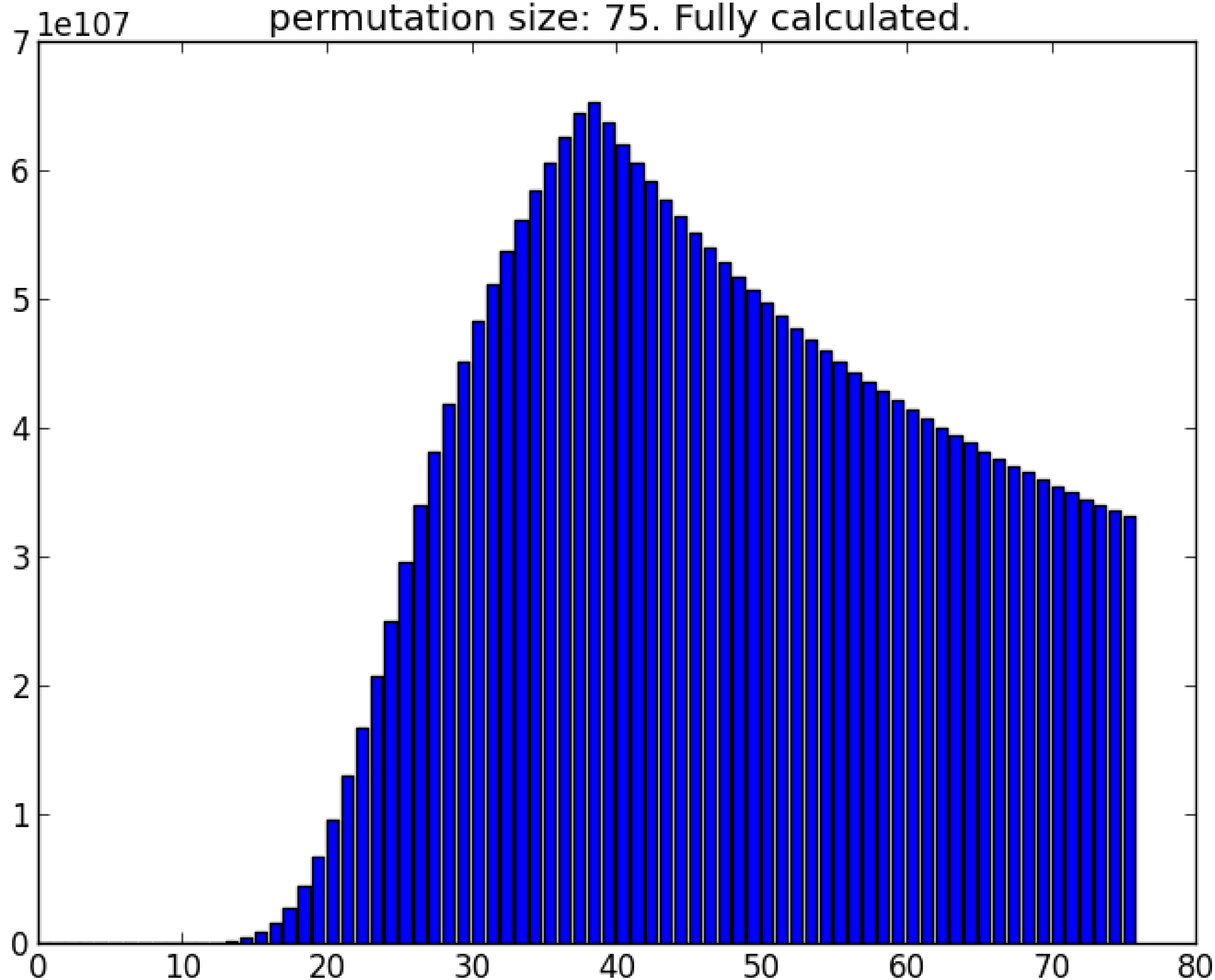

perm size: 8, num_perms=8!
9000
8000
7000
6000
5000
4000
3000
2000
1000
0
1
2
3
4
5
6
7
8
9

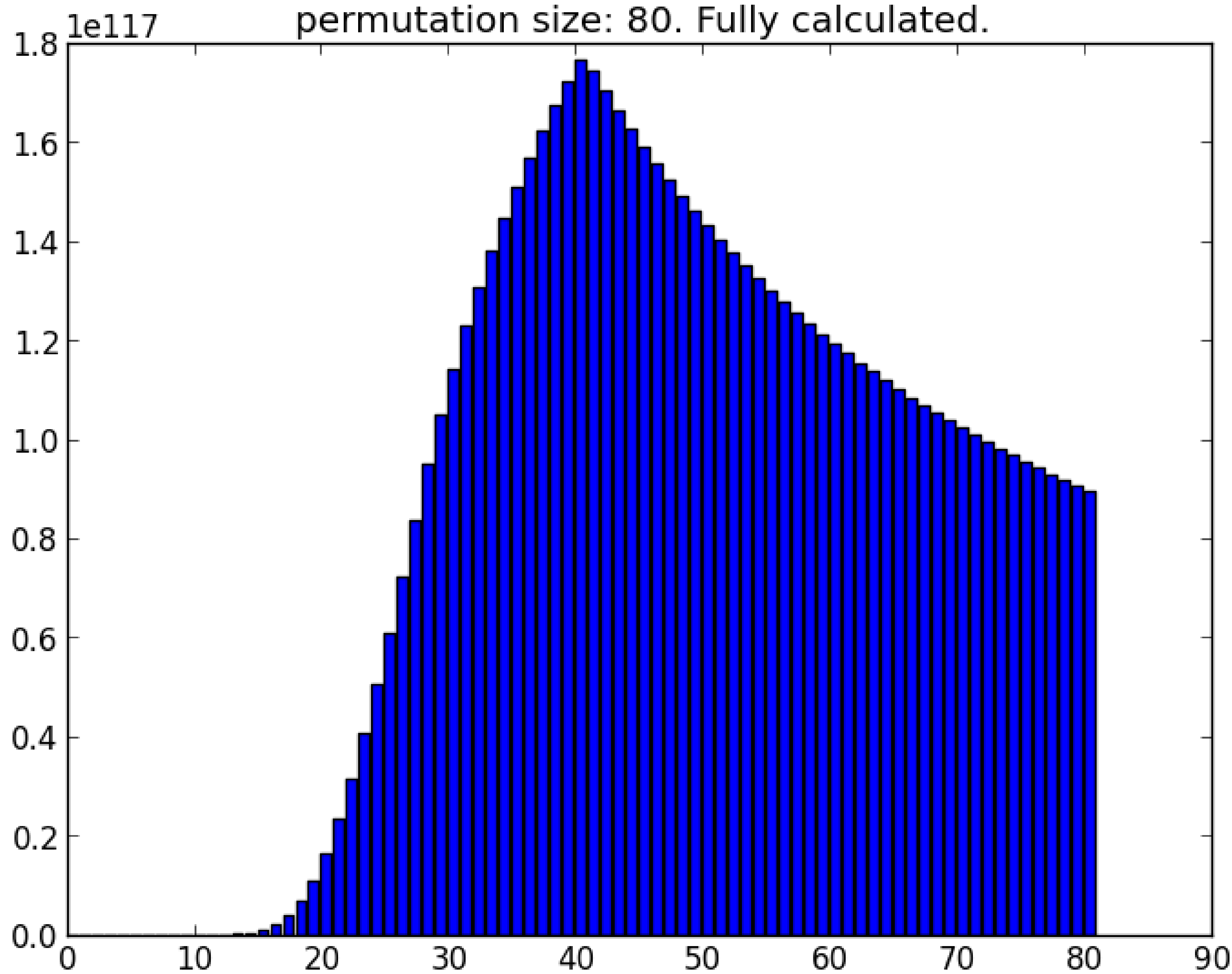
n=9 case


perm size: 9, num_perms=9!
80000
70000
60000
50000
40000
30000
20000
10000
0
1
2
3
4
5
6
7
8
9
10

n=10 case

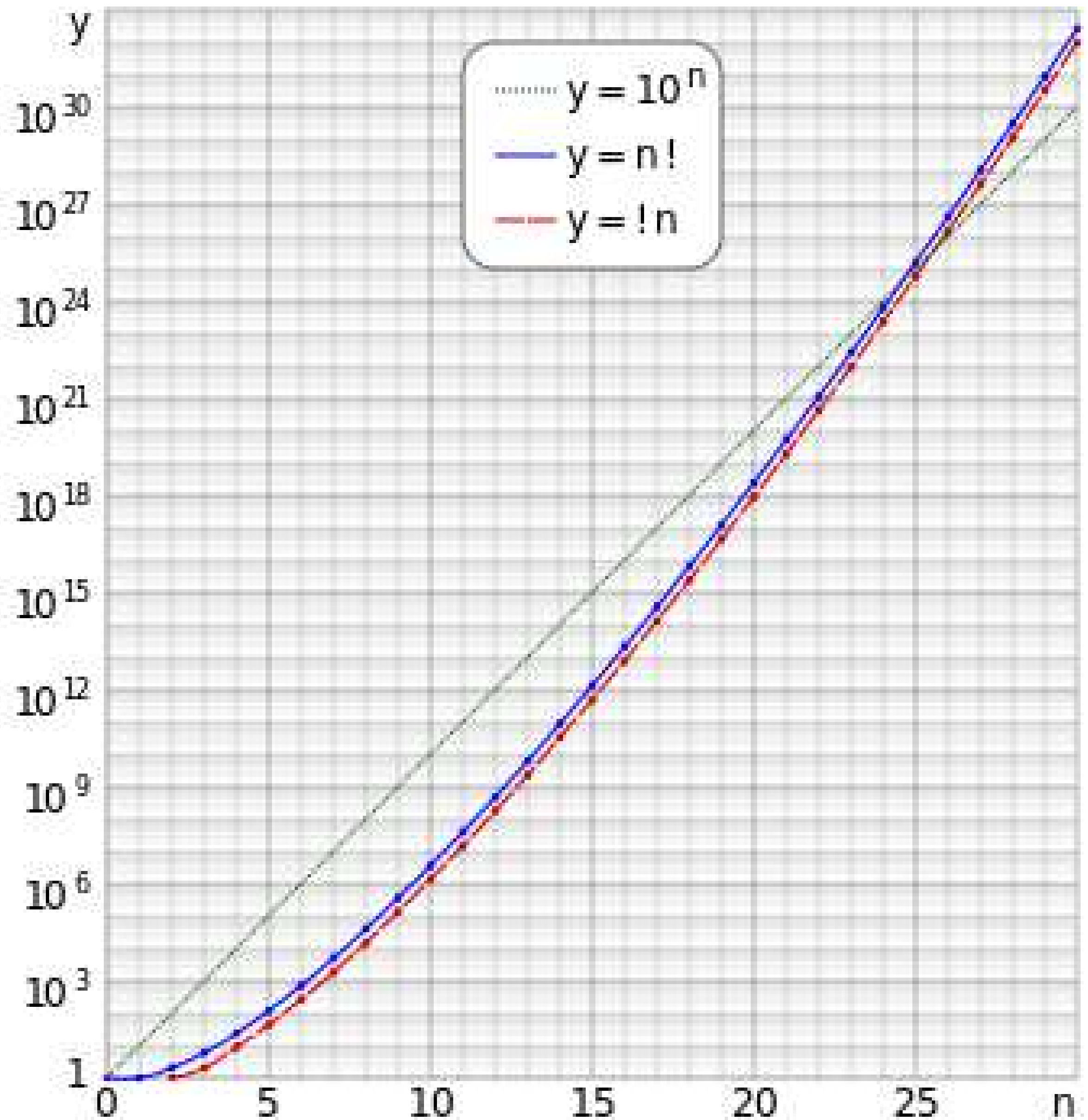

perm size: 10, num_perms=10!
700000
600000
500000
400000
300000
200000
100000
0
0
2
4
6
8
10
12


permutation size: 75. Fully calculated.
1e107
0
1
2
3
4
5
6
7
0
10
20
30
40
50
60
70
80

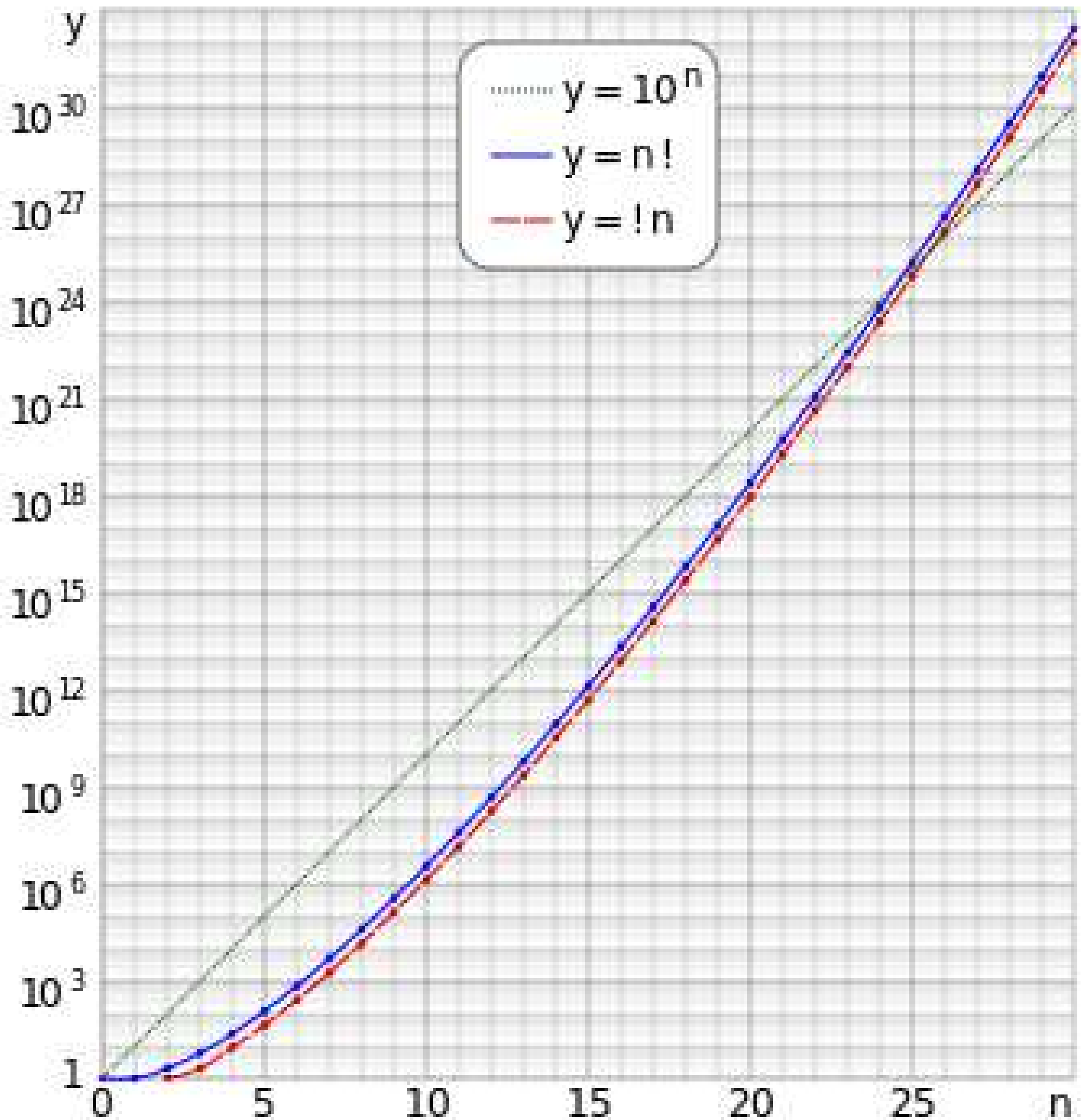


permutation size: 80. Fully calculated.
1e117
1.8
1.6
1.4
1.2
1.0
0.8
0.6
0.4
0.2
0.0
0
10
20
30
40
50
60
70
80
90

How many permuations (n!) are derangements (!n) ?

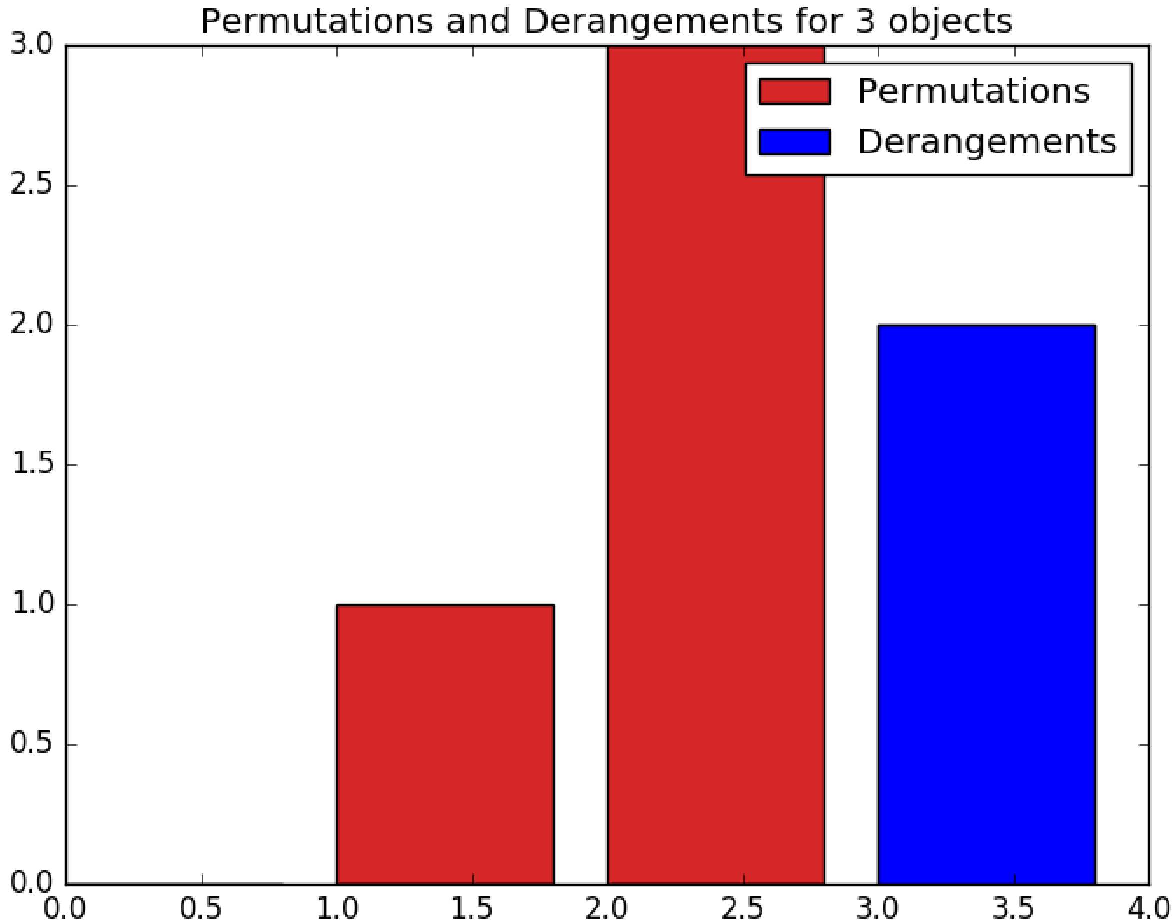


How many permuations (n!) are derangements (!n) ?

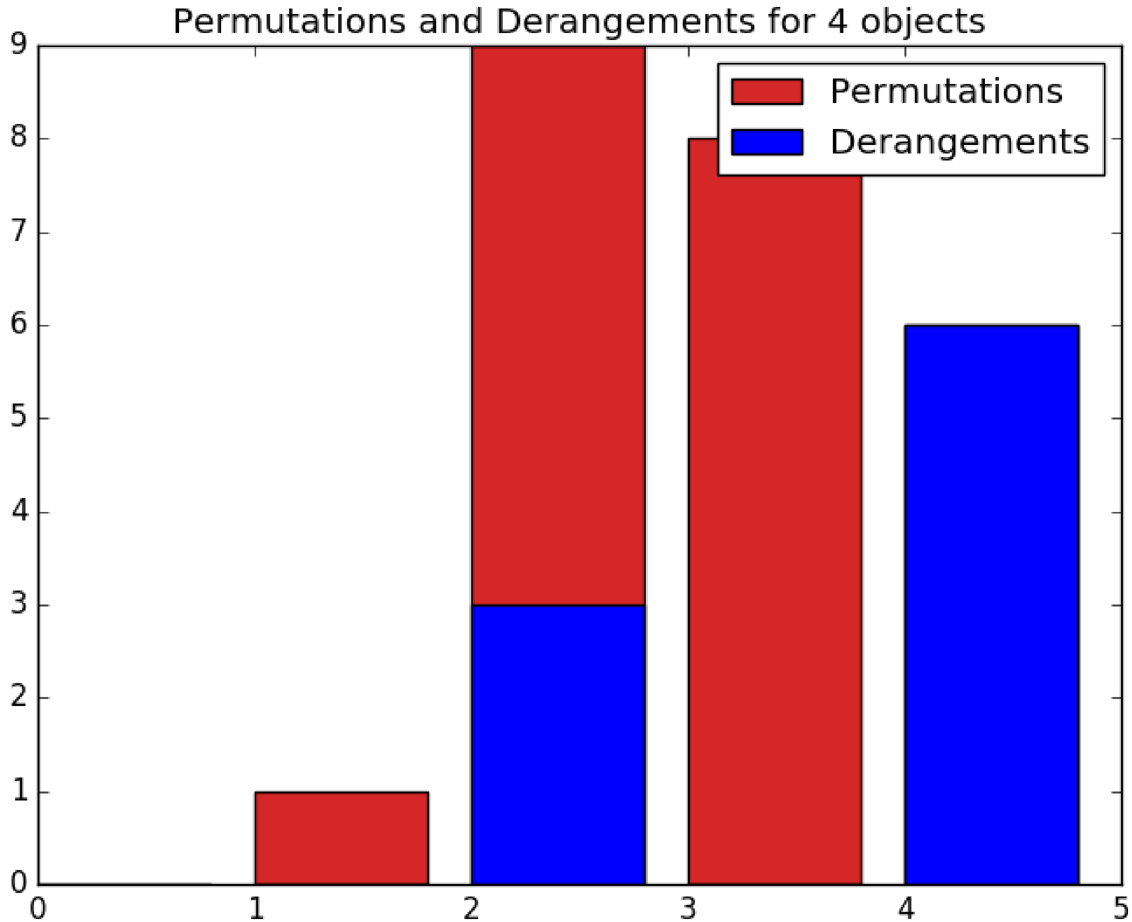


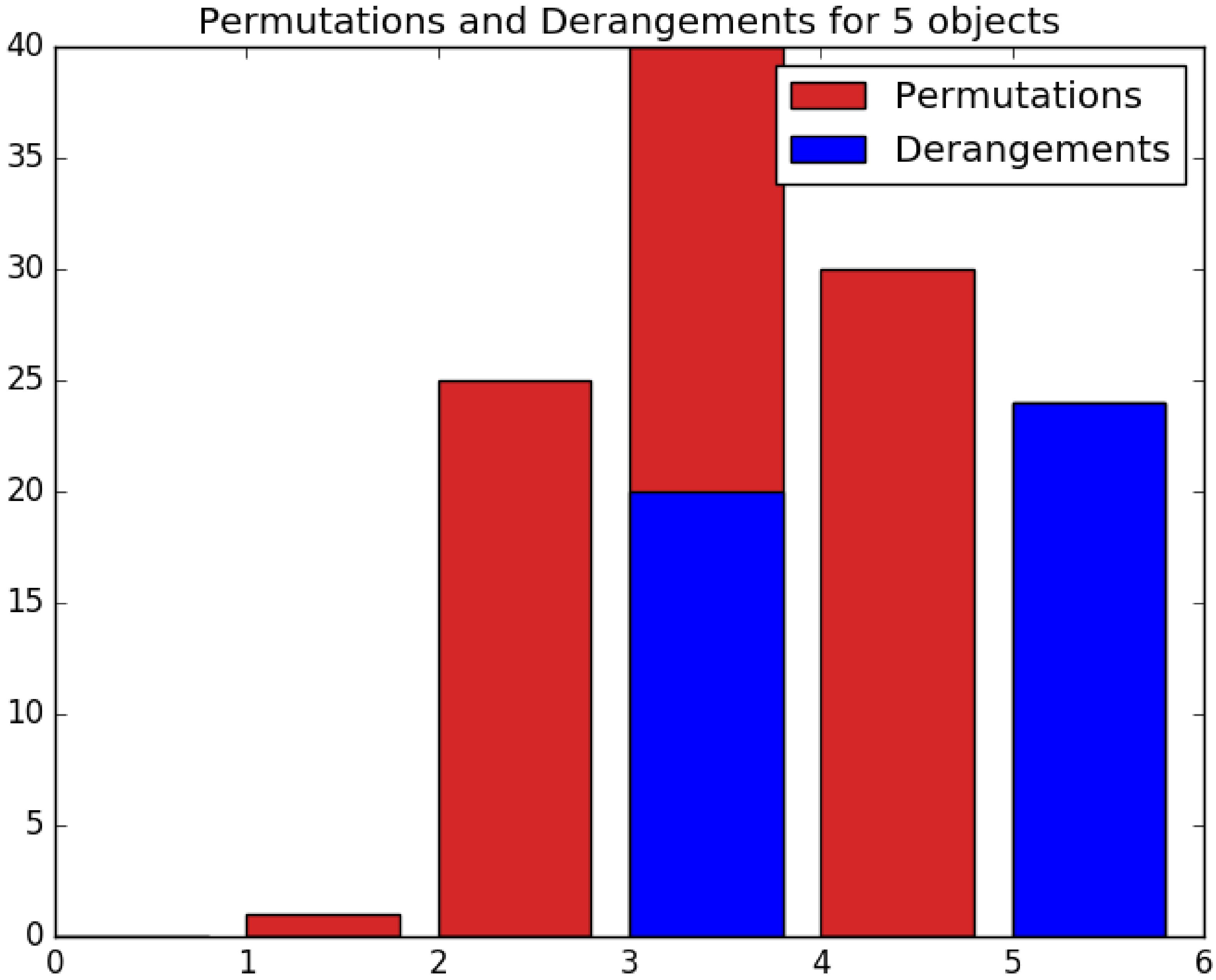
1/e of the permutations are derangements (Thanks to Jim Blowers for pointing this out!)

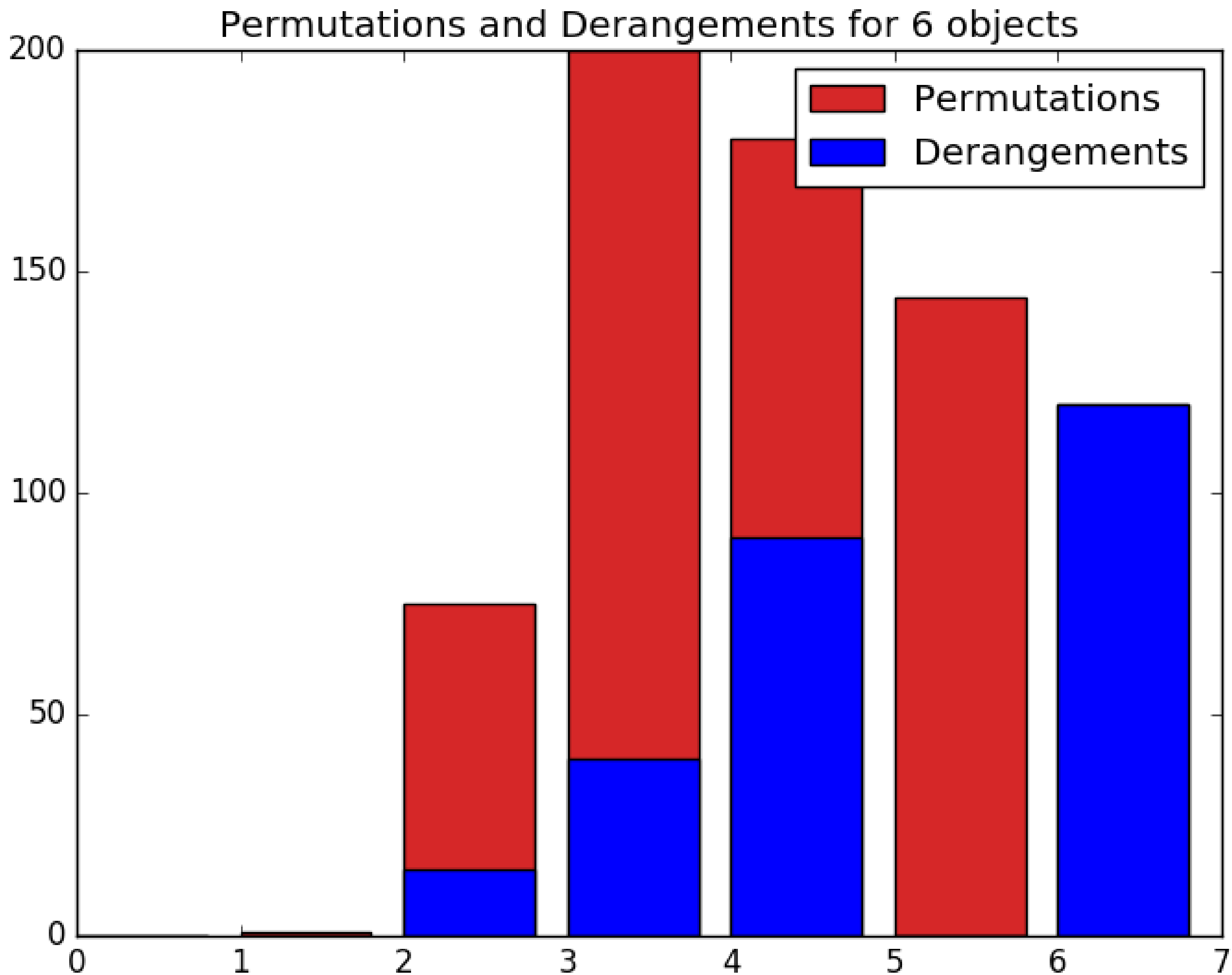
n=3: 2 of 6 are derangements

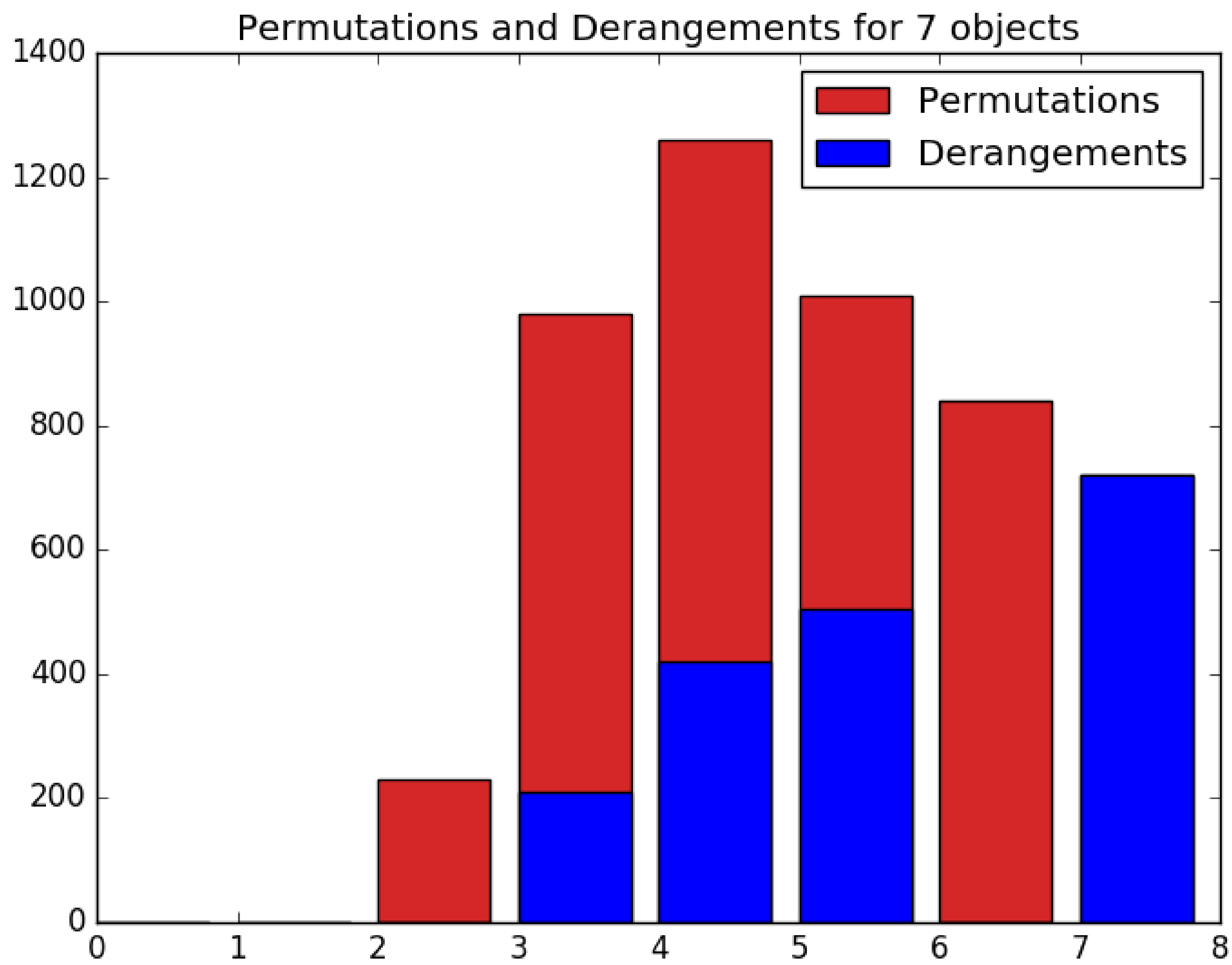


n=4: 9 of 24 are derangements

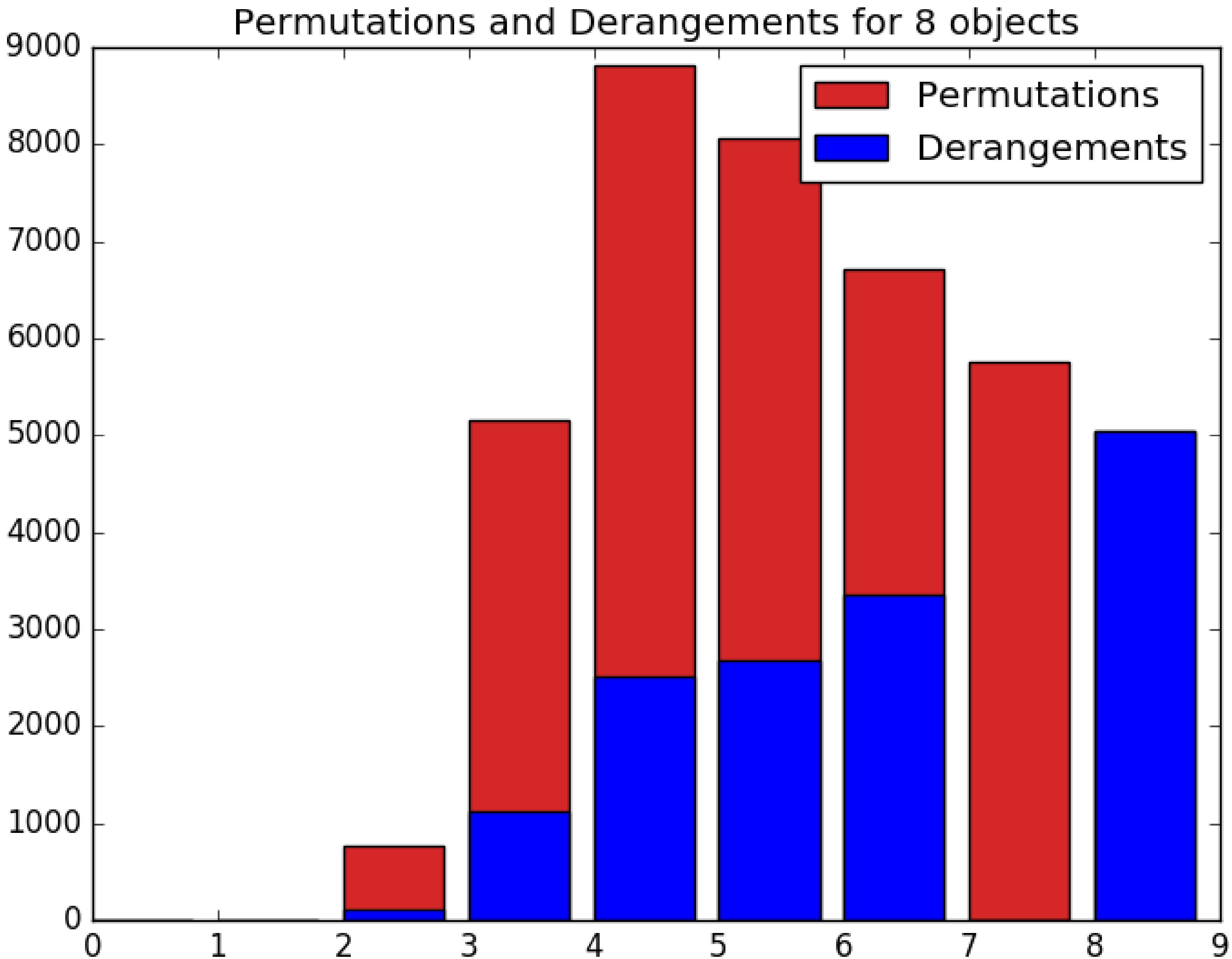


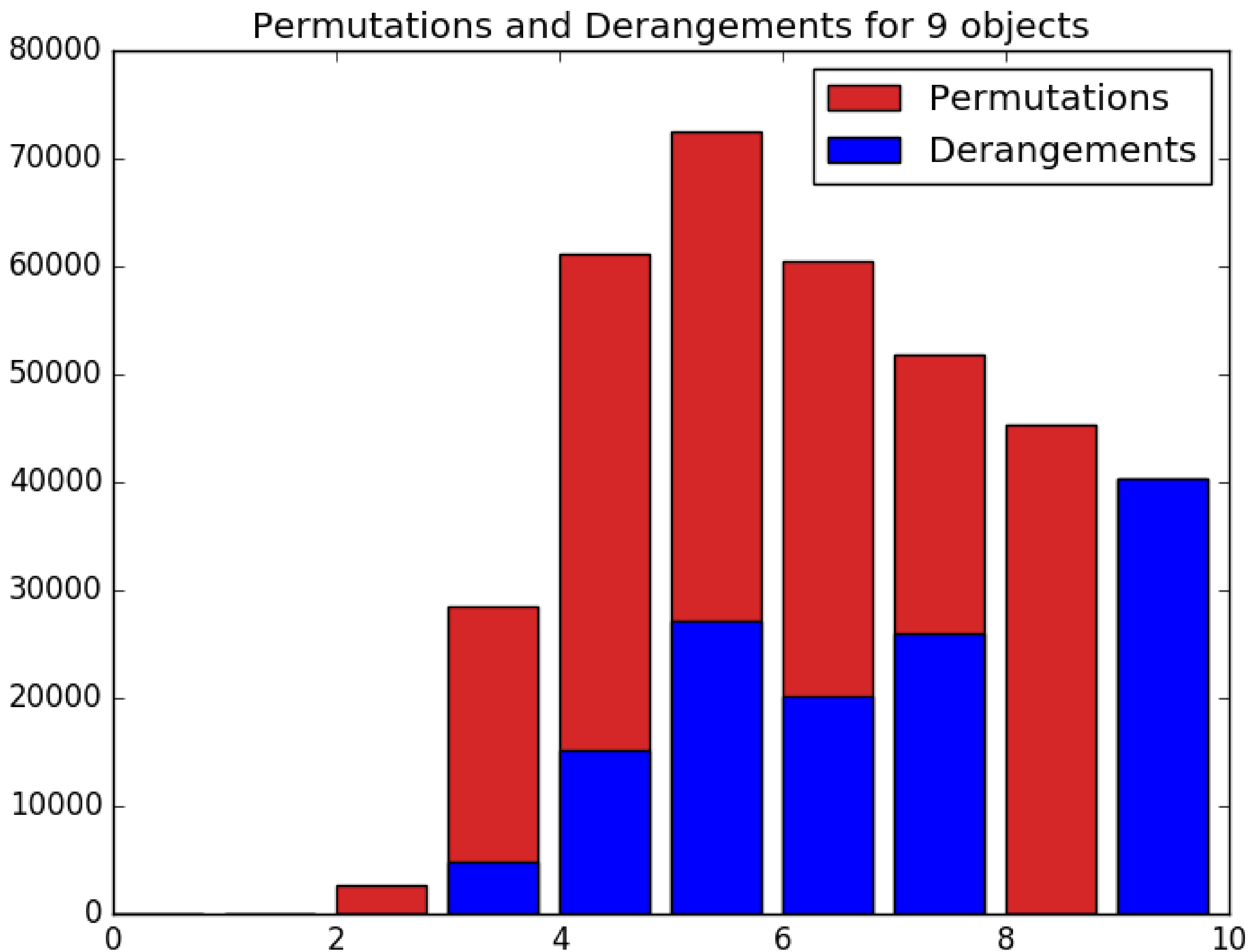

Permutations and Derangements for 5 objects
Permutations
Derangements
0
5
10
15
20
25
30
35
40
0
1
2
3
4
5
6

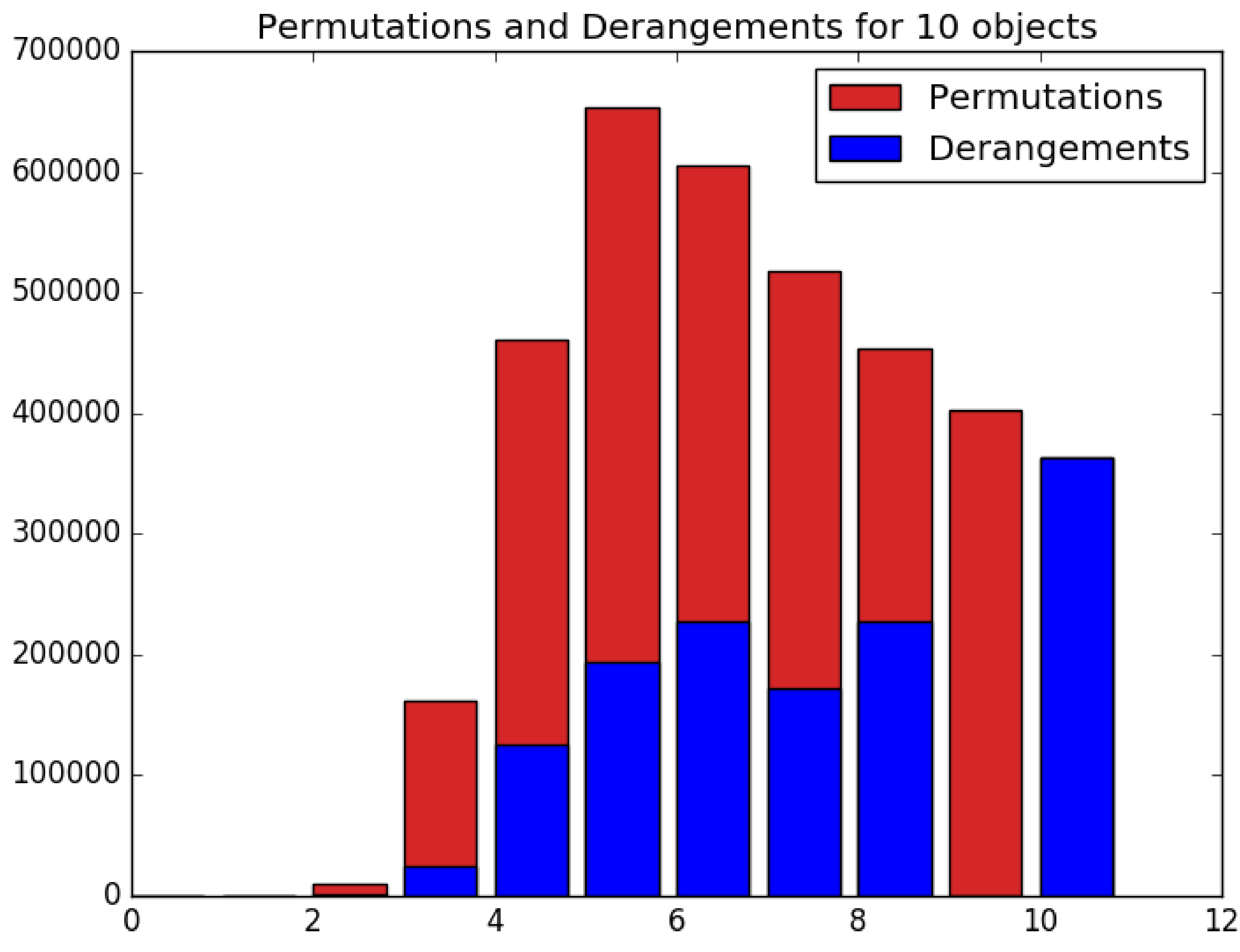

Permutations and Derangements for 6 objects
Permutations
Derangements
0
50
100
150
200
0
1
2
3
4
5
6
7



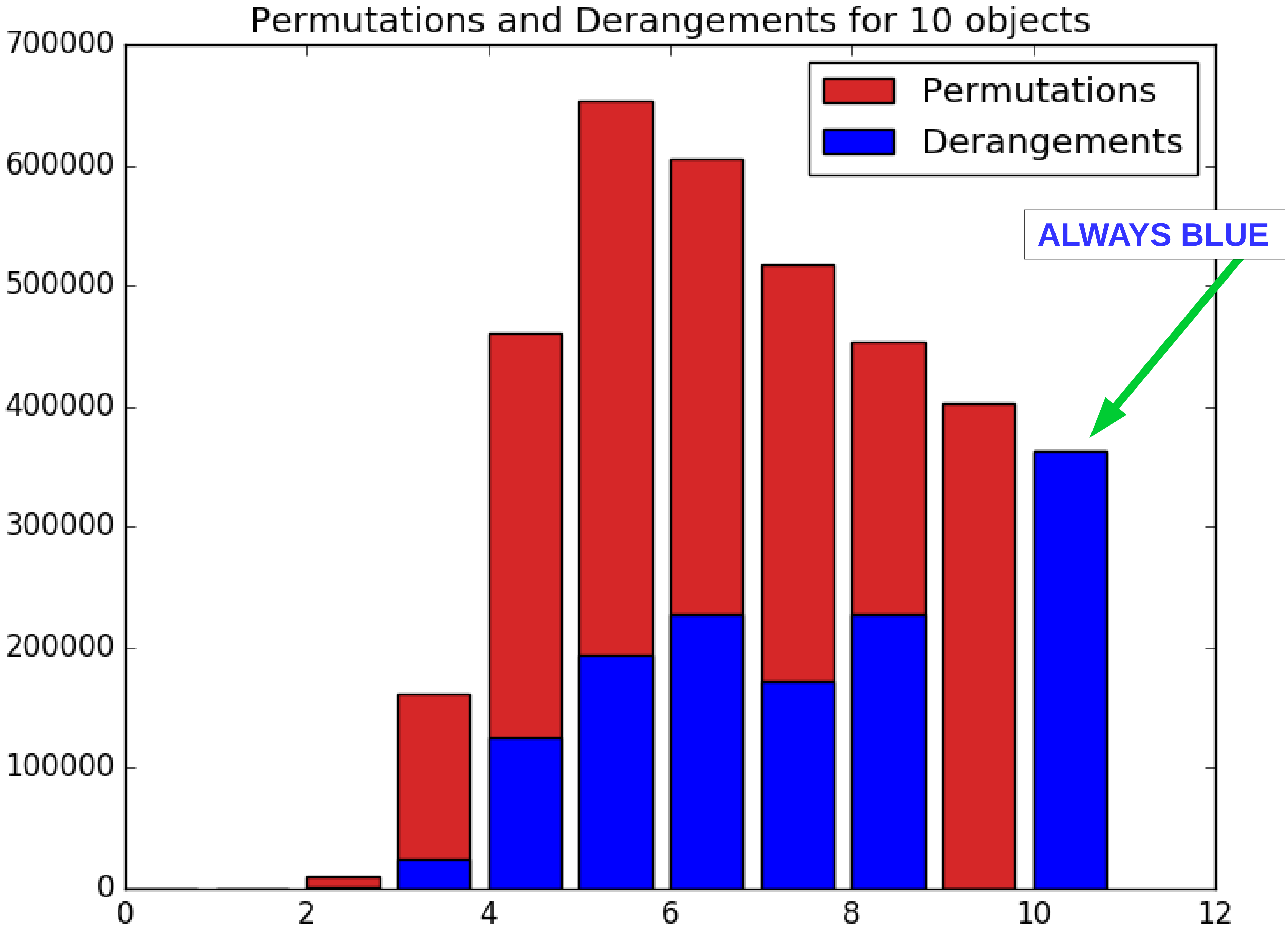
Permutations and Derangements for 7 objects
Permutations
Derangements
1400
1200
1000
800
600
400
200
0
0
1
2
3
4
5
6
7
8

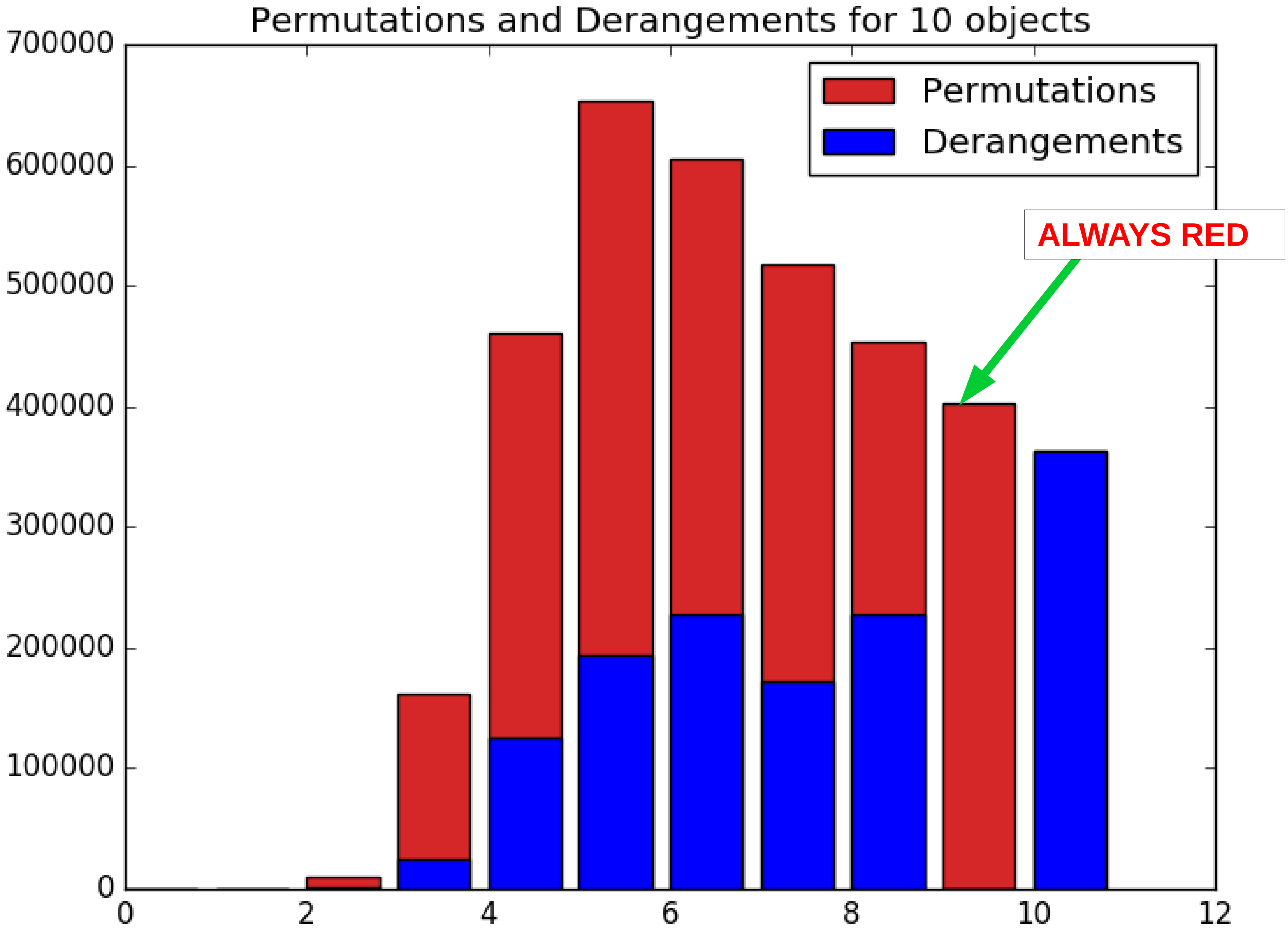

Permutations and Derangements for 8 objects
Permutations
Derangements
9000
8000
7000
6000
5000
4000
3000
2000
1000
0
0
1
2
3
4
5
6
7
8
9

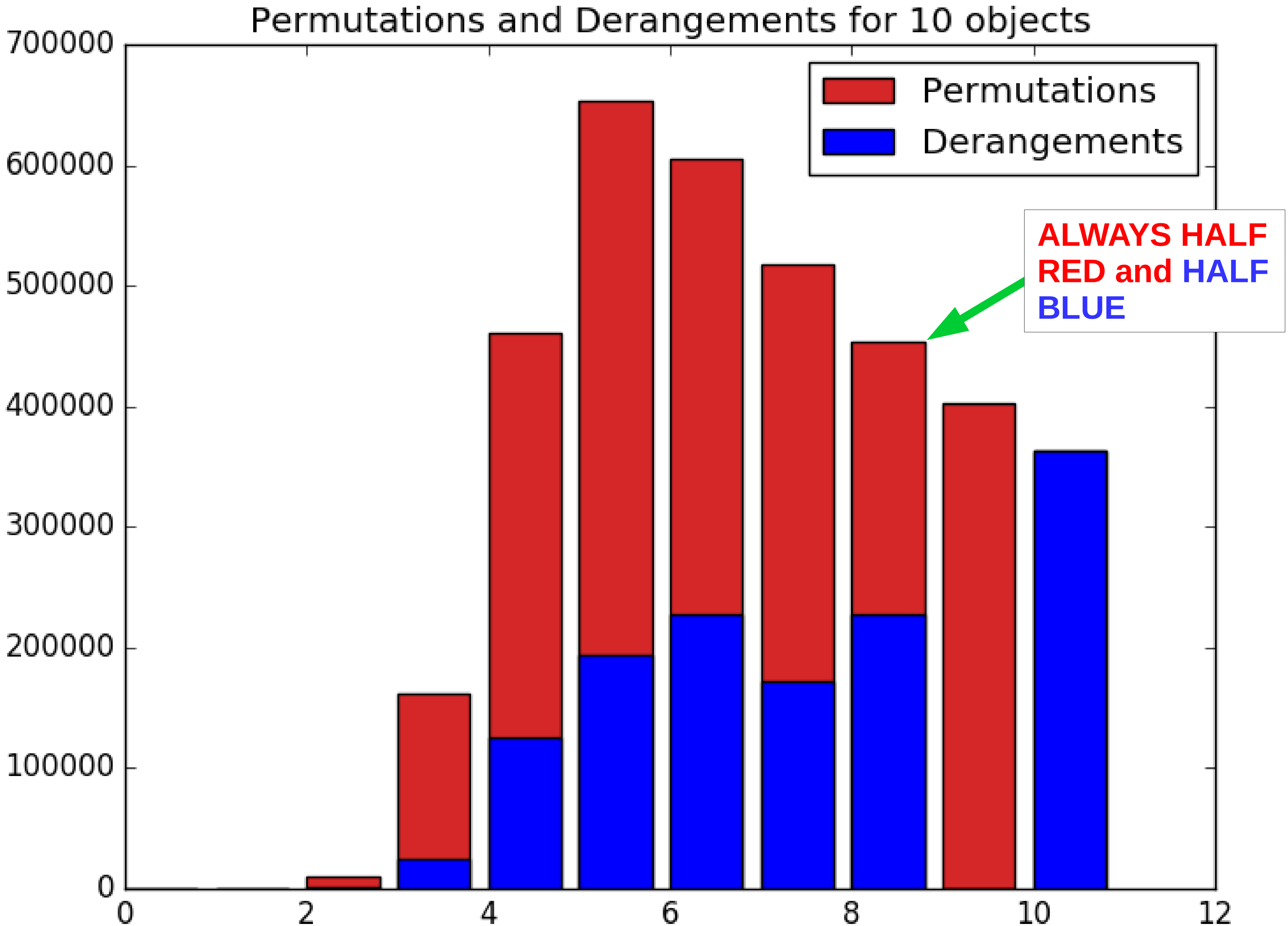

Permutations and Derangements for 9 objects
Permutations
Derangements
80000
70000
60000
50000
40000
30000
20000
10000
0
0
2
4
6
8
10

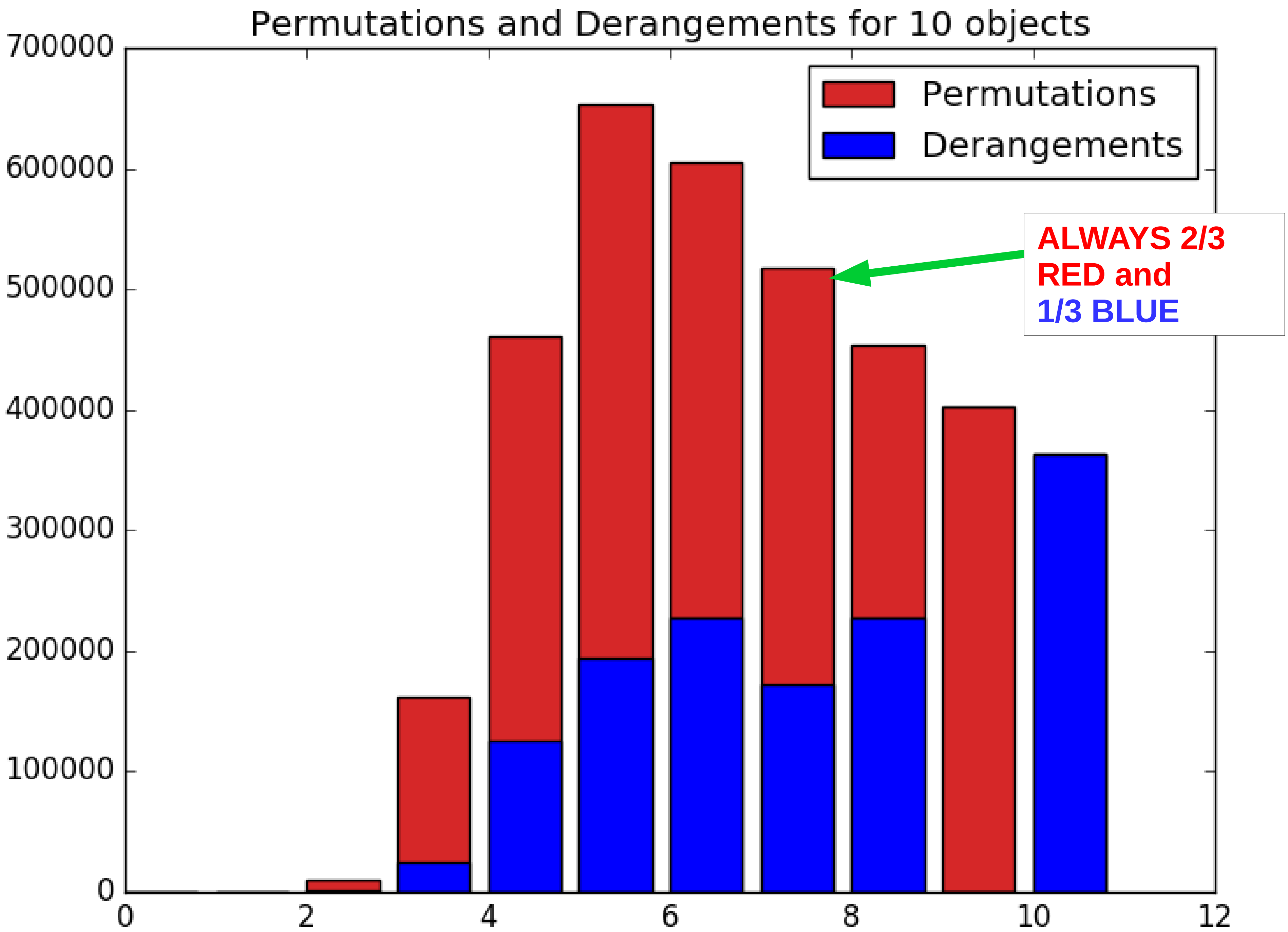


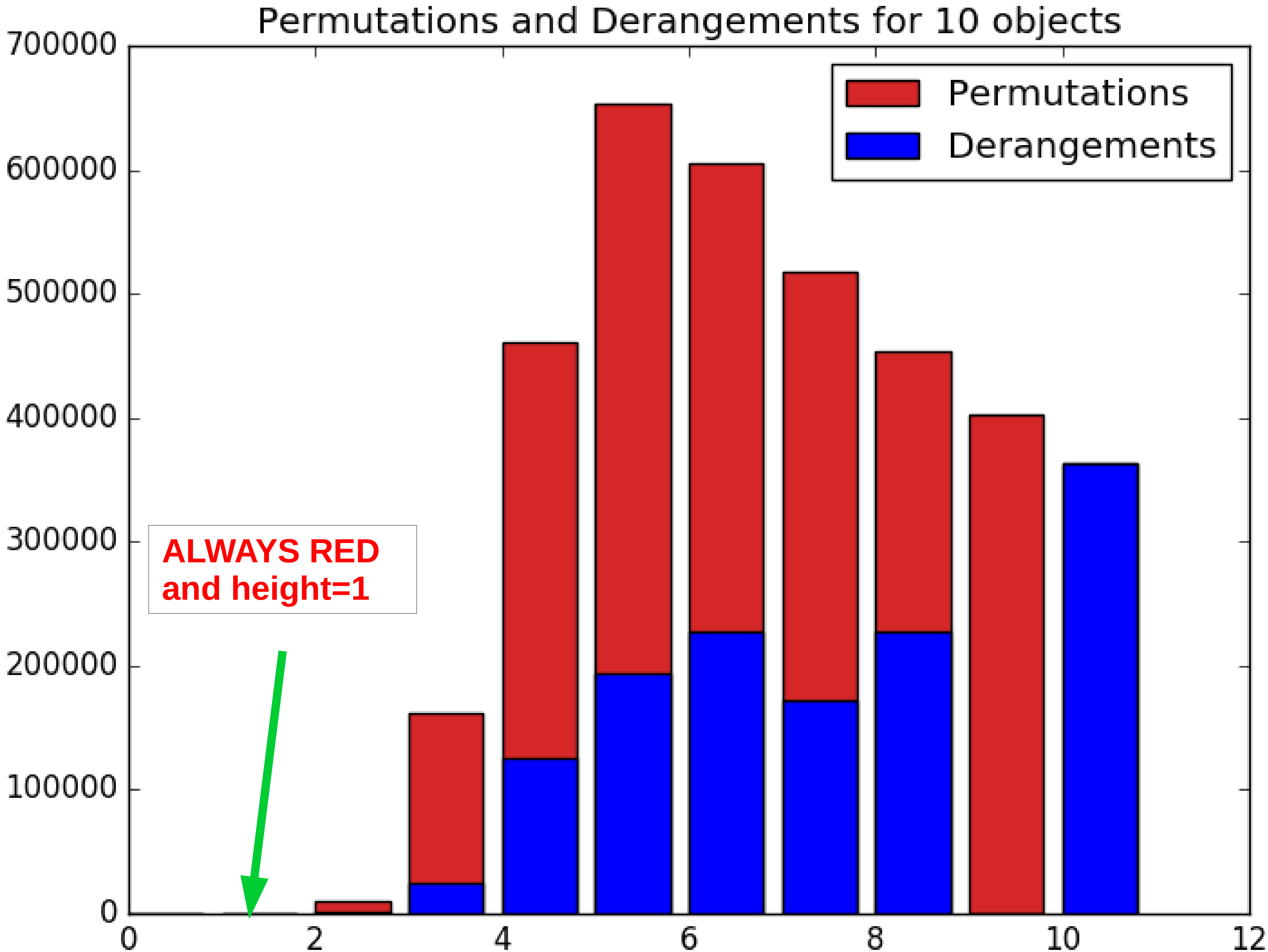
Permutations and Derangements for 10 objects
Permutations
Derangements
700000
600000
500000
400000
300000
200000
100000
0
0
2
4
6
8
10
12

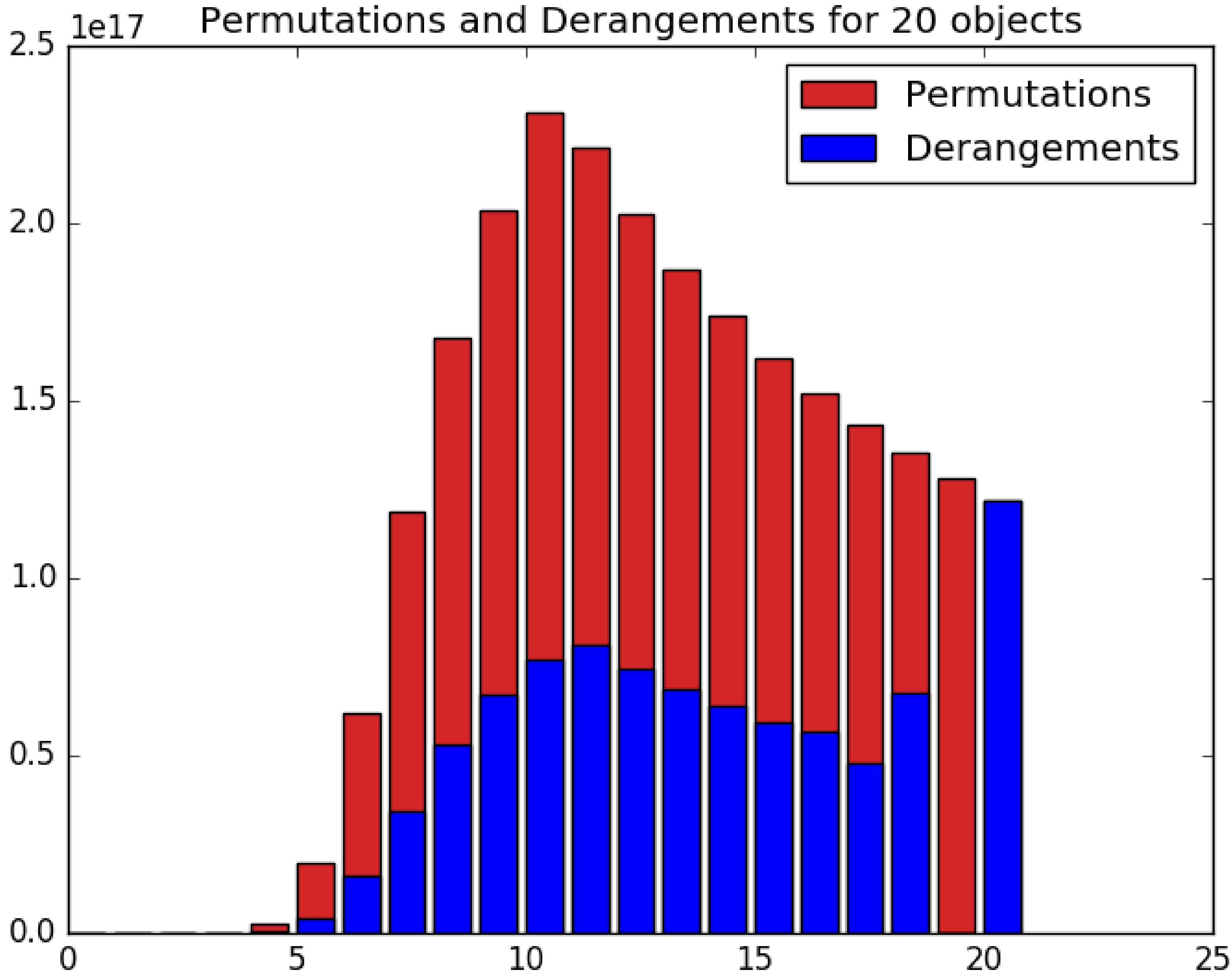

Permutations and Derangements for 10 objects
Permutations
Derangements
ALWAYS BLUE
0
100000
200000
300000
400000
500000
600000
700000
0
2
4
6
8
10
12


Permutations and Derangements for 10 objects
Permutations
Derangements
ALWAYS RED
700000
600000
500000
400000
300000
200000
100000
0
0
2
4
6
8
10
12

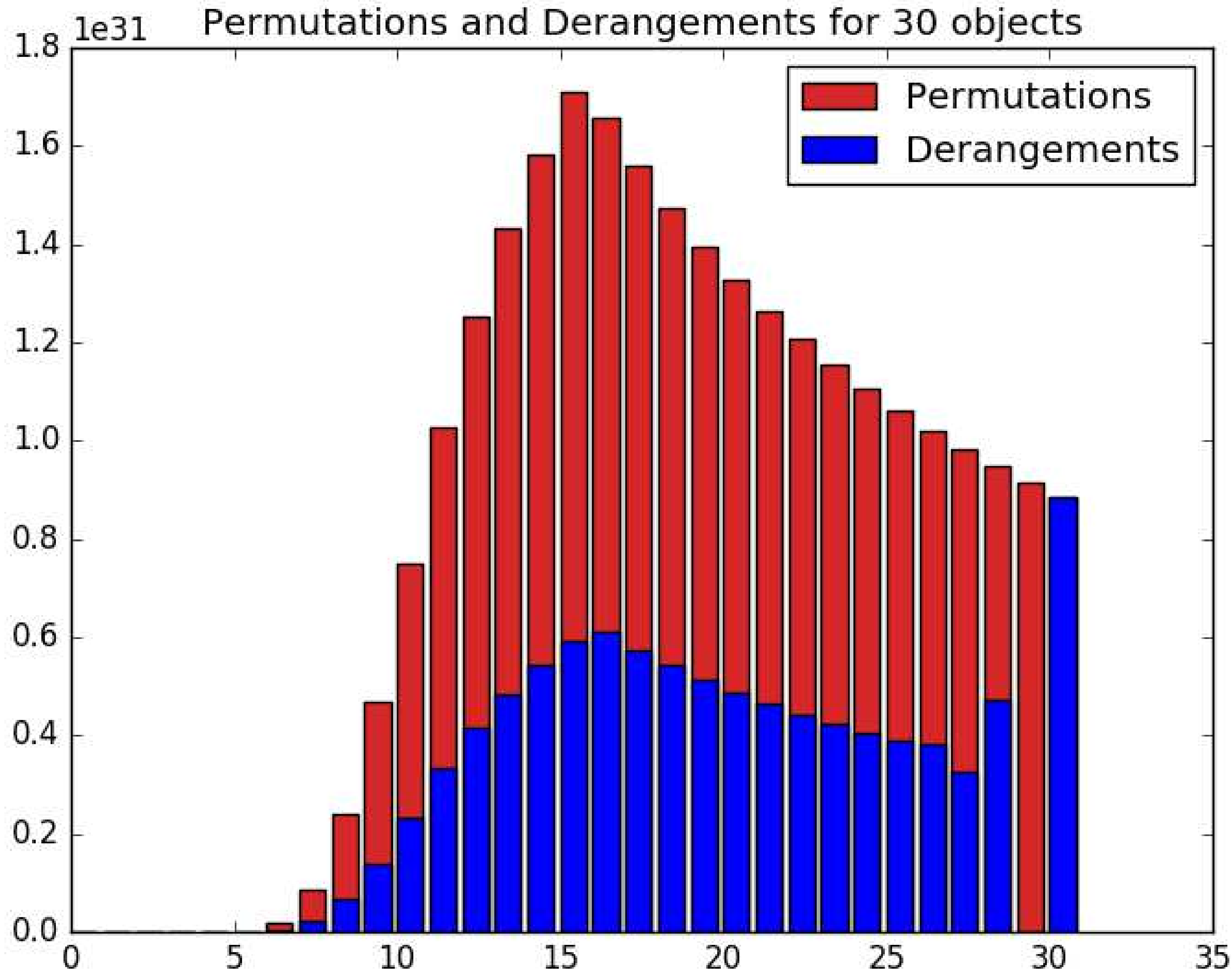

Permutations and Derangements for 10 objects
Permutations
Derangements
ALWAYS HALF RED and HALF BLUE
0
100000
200000
300000
400000
500000
600000
700000
0
2
4
6
8
10
12


Permutations and Derangements for 10 objects
700000
600000
500000
400000
300000
200000
100000
0
0
2
4
6
8
10
12
Permutations
Derangements
ALWAYS 2/3
RED and
1/3 BLUE

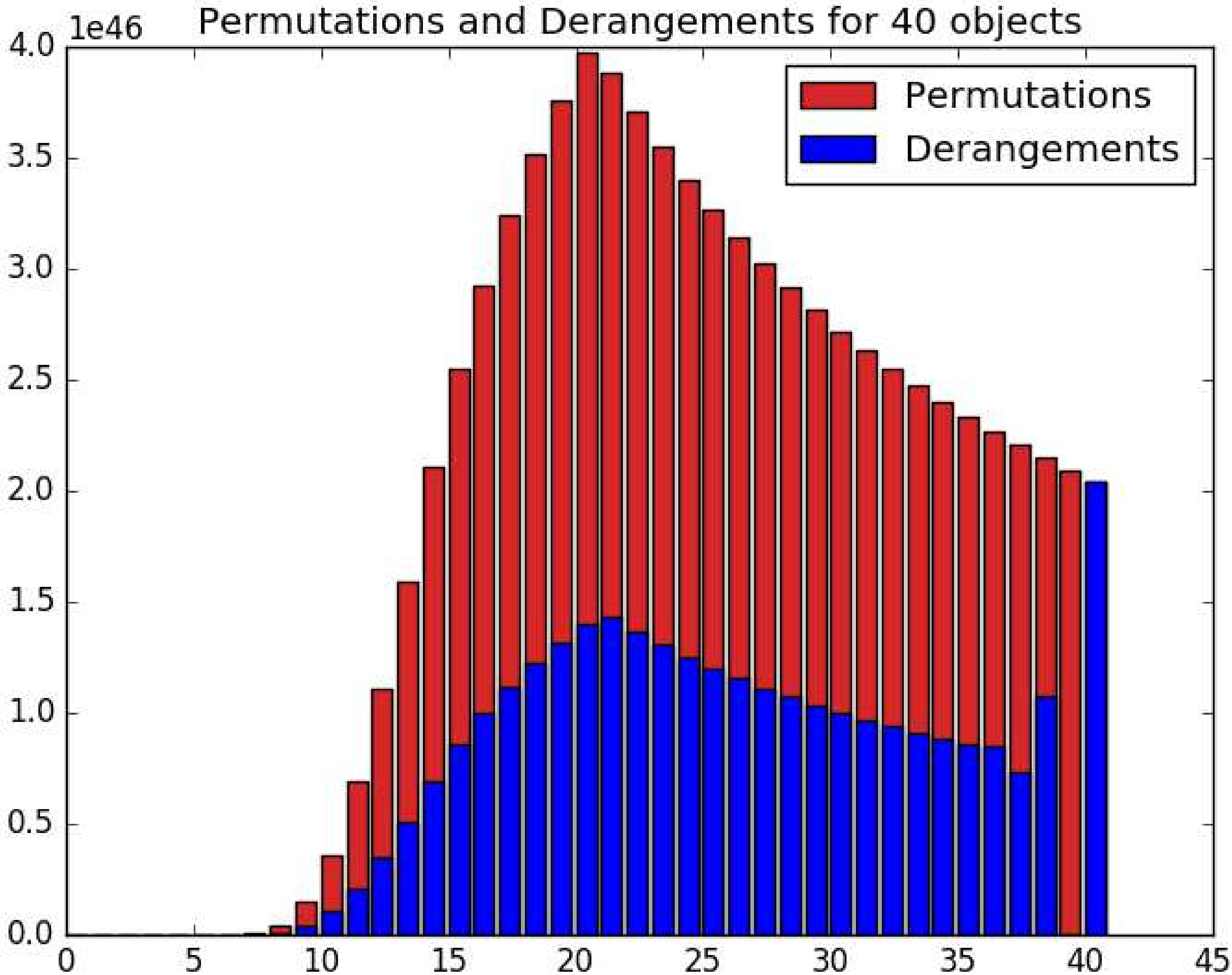

Permutations and Derangements for 10 objects
700000
600000
500000
400000
300000
200000
100000
0
0
2
4
6
8
10
12
Permutations
Derangements
ALWAYS RED
and height=1

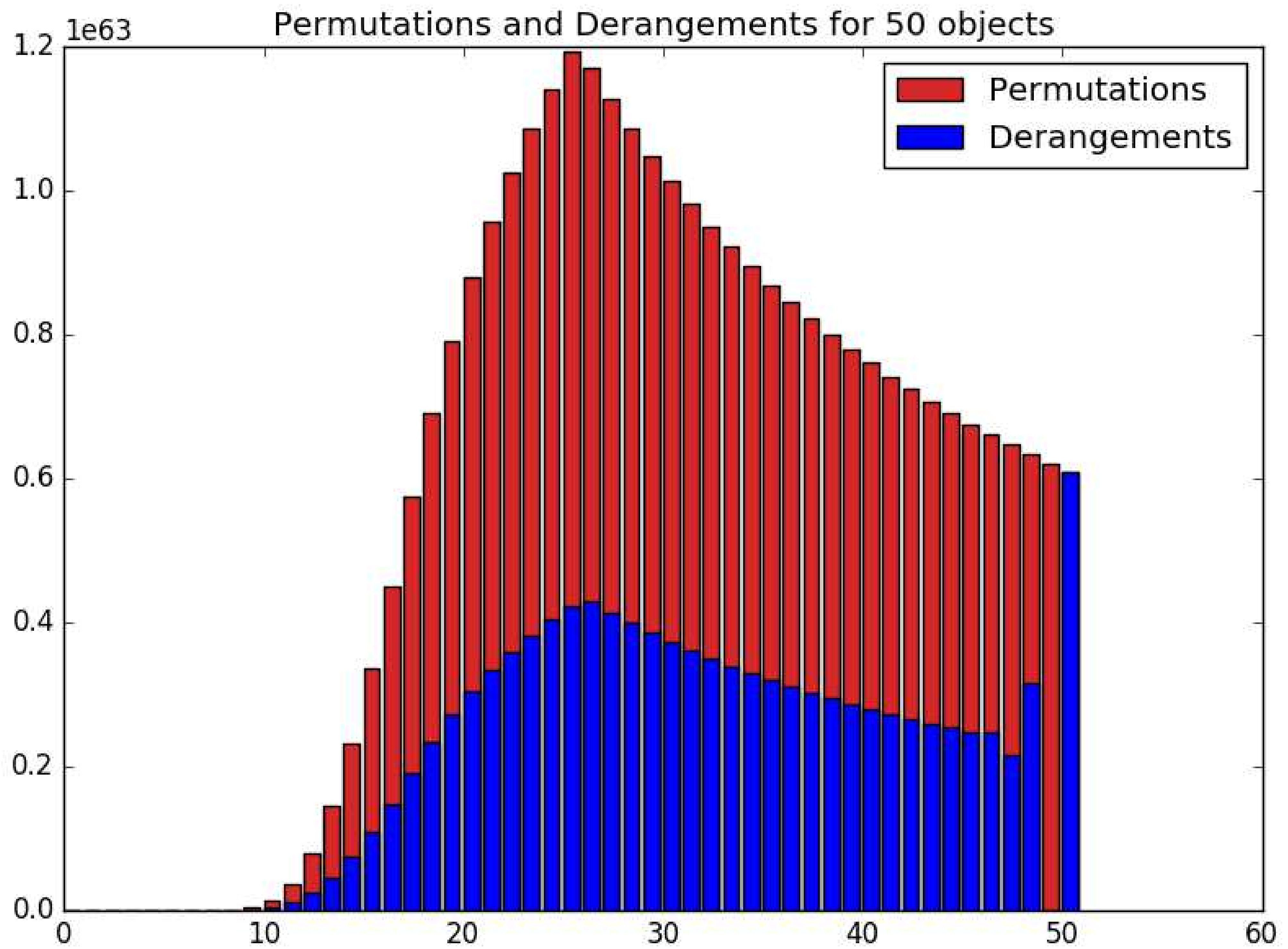


Permutations and Derangements for 20 objects
1e17
2.5
2.0
1.5
1.0
0.5
0.0
0
5
10
15
20
25
Permutations
Derangements



Permutations and Derangements for 30 objects
1e31
Permutations
Derangements
0.0
0.2
0.4
0.6
0.8
1.0
1.2
1.4
1.6
1.8
0
5
10
15
20
25
30
35


Permutations and Derangements for 40 objects
1e46
4.0
3.5
3.0
2.5
2.0
1.5
1.0
0.5
0.0
0
5
10
15
20
25
30
35
40
45
Permutations
Derangements



Permutations and Derangements for 50 objects
1e63
1.2
1.0
0.8
0.6
0.4
0.2
0.0
0
10
20
30
40
50
60
Permutations
Derangements

Second Game: Mom's Ice-Breaker

- Someone starts the introductions
- It has to be a derangement
- But what do you think happened when we got back to the first person?

Second Game: Mom's Ice-Breaker

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  - Ok, I guess we're all finished!
  - On to the next activity!
  - Oh wait! We left people out!

Second Game: Mom's Ice-Breaker

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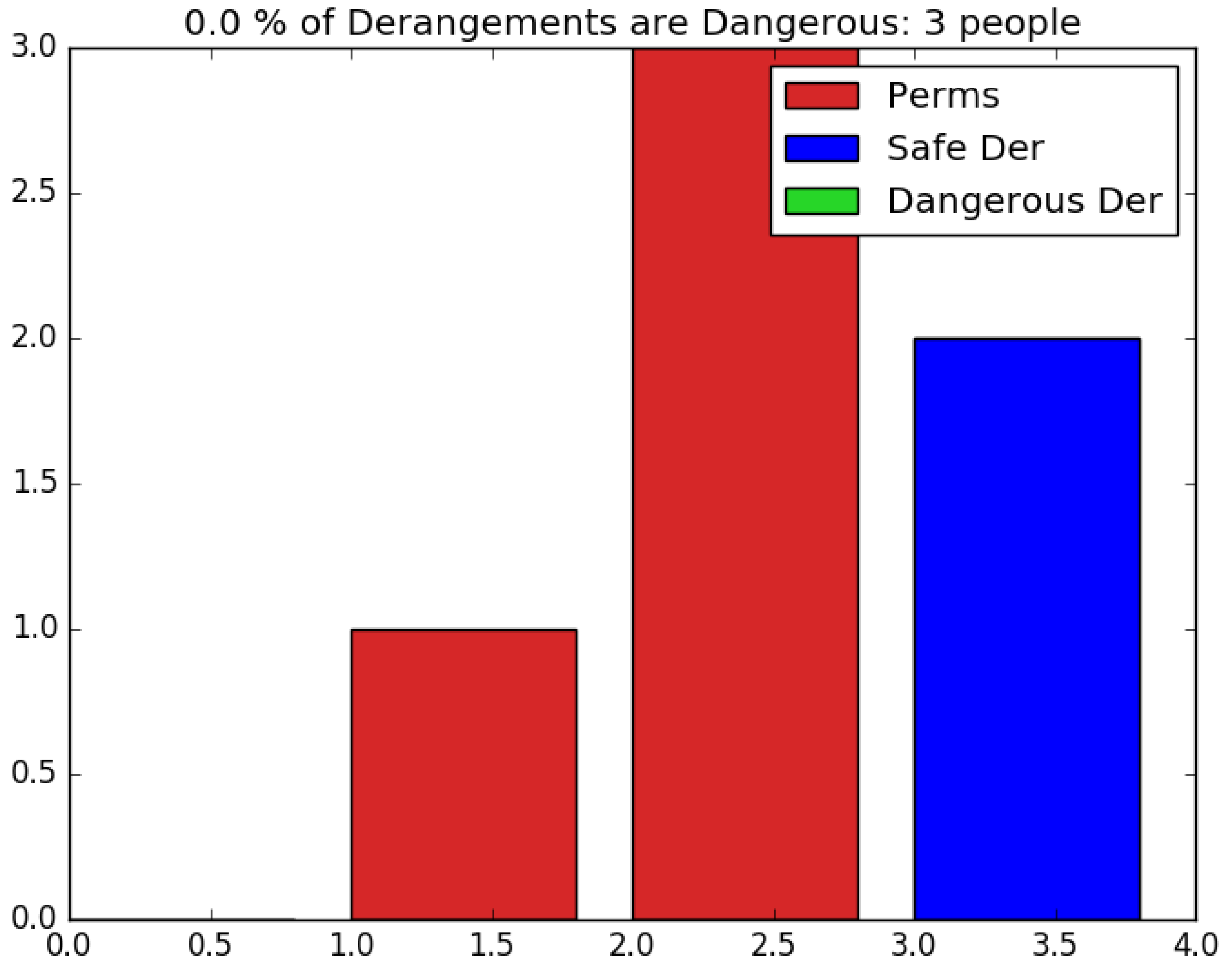
- It has to be a derangement
- But what do you think happened when we got back to the first person?
  - Ok, I guess we're all finished!
  - On to the next activity!
  - Oh wait! We left people out!

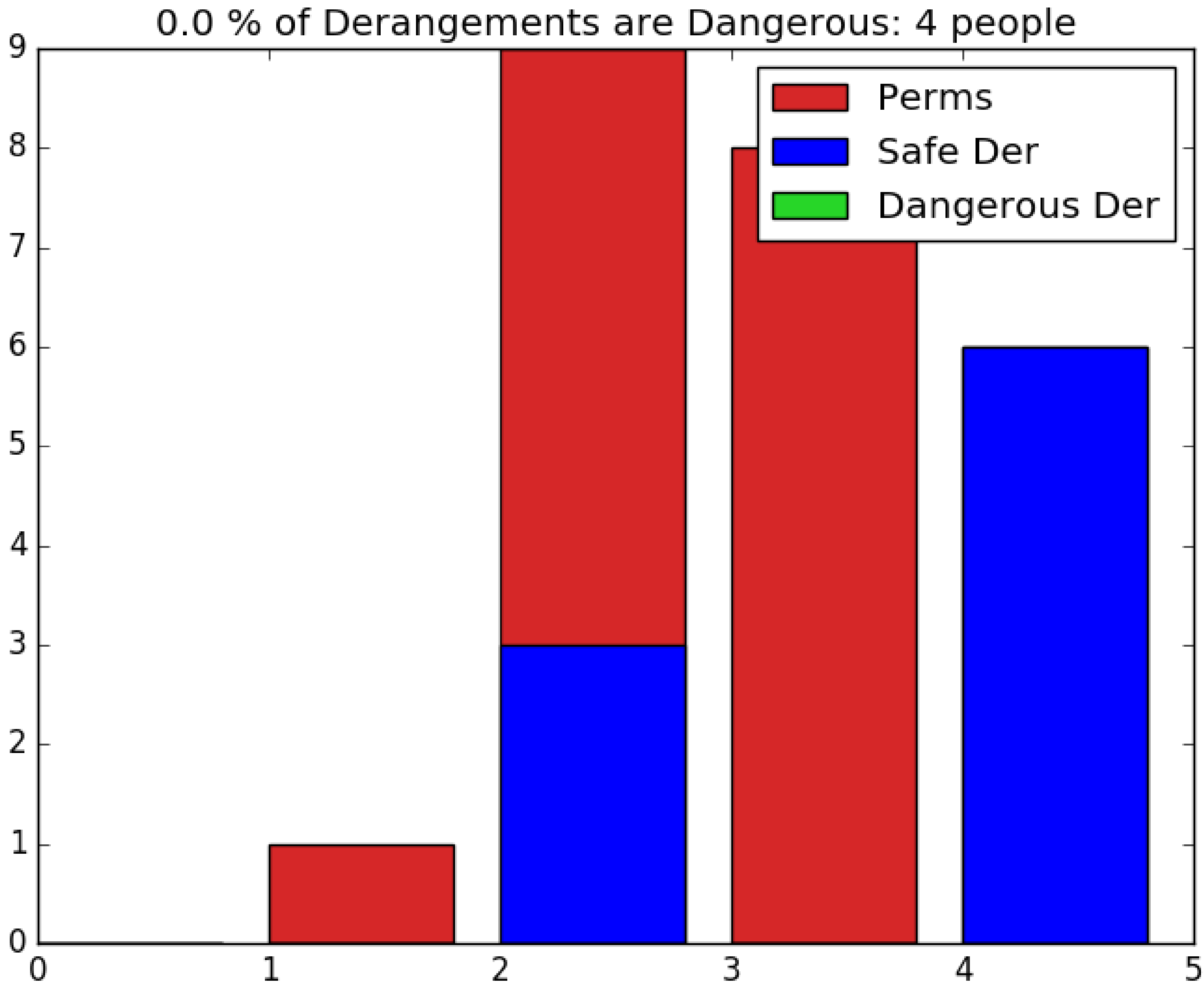
AWKWARD!!!

What is socially dangerous?

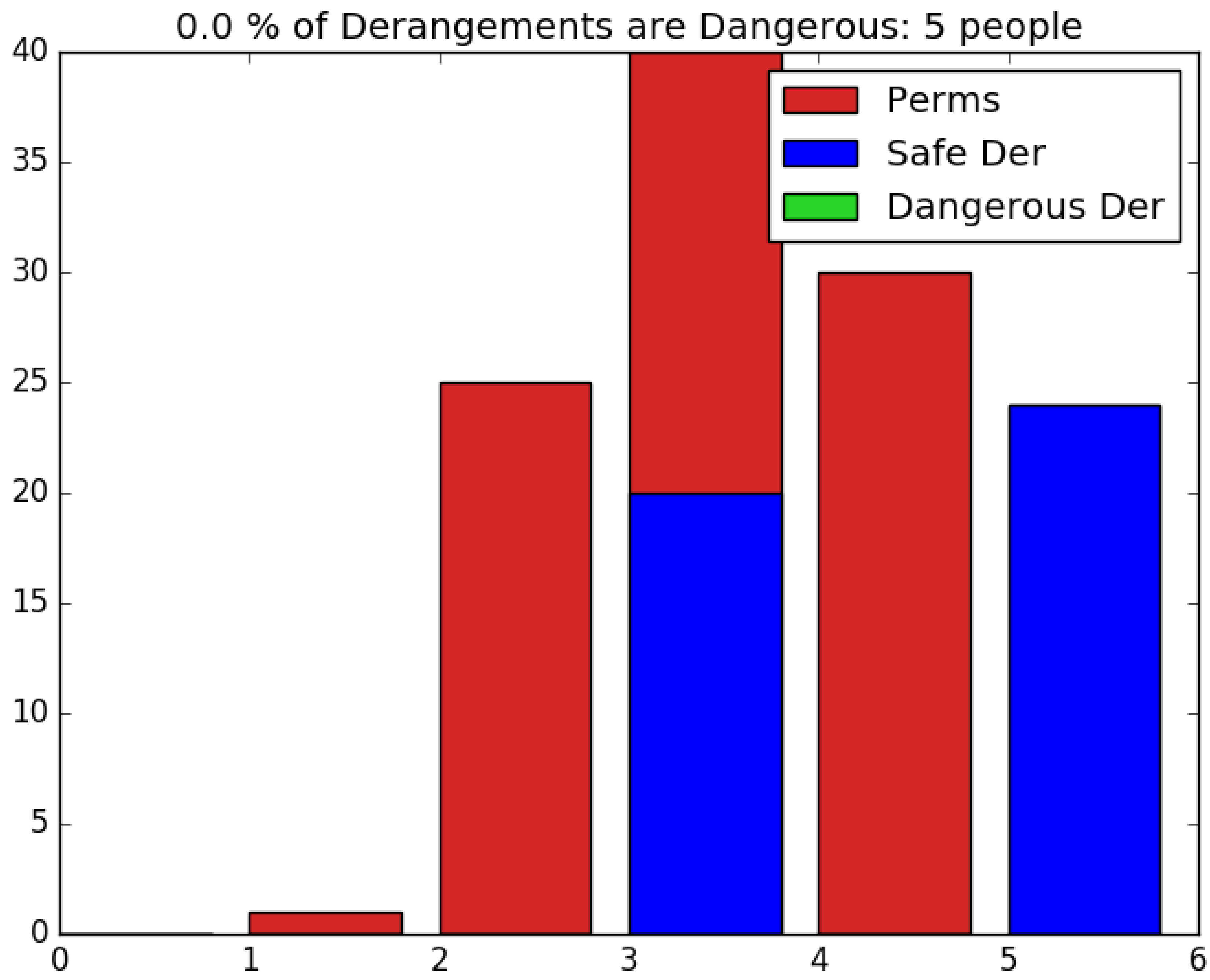
- My heuristic guess:
  - Everyone in a single cycle is fine
  - Lots of small cycles is also fine
  - If the largest cycle has over 2/3rds of the people in it, the remaining participants are likely to be forgotten.

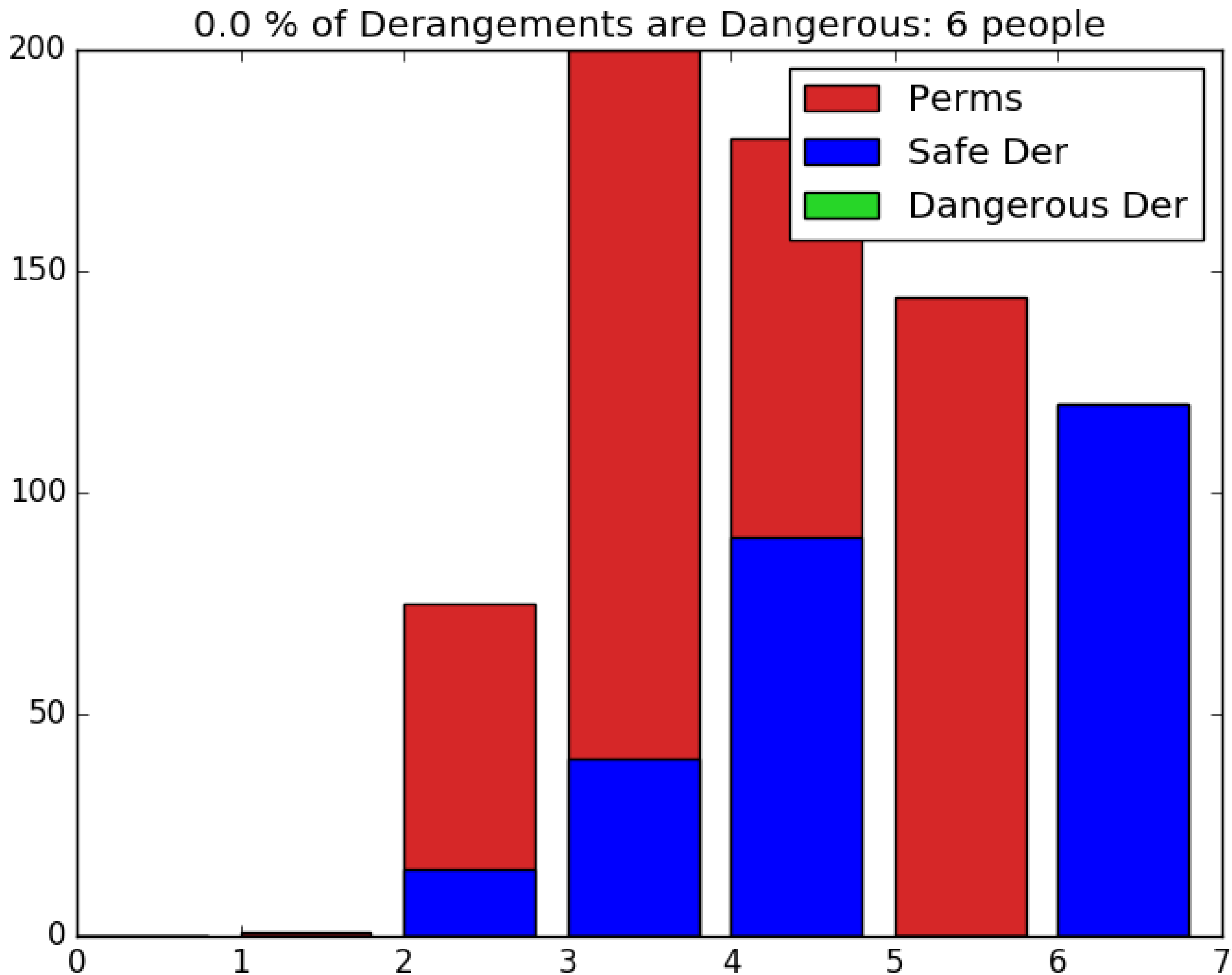
No such danger in group of 3.

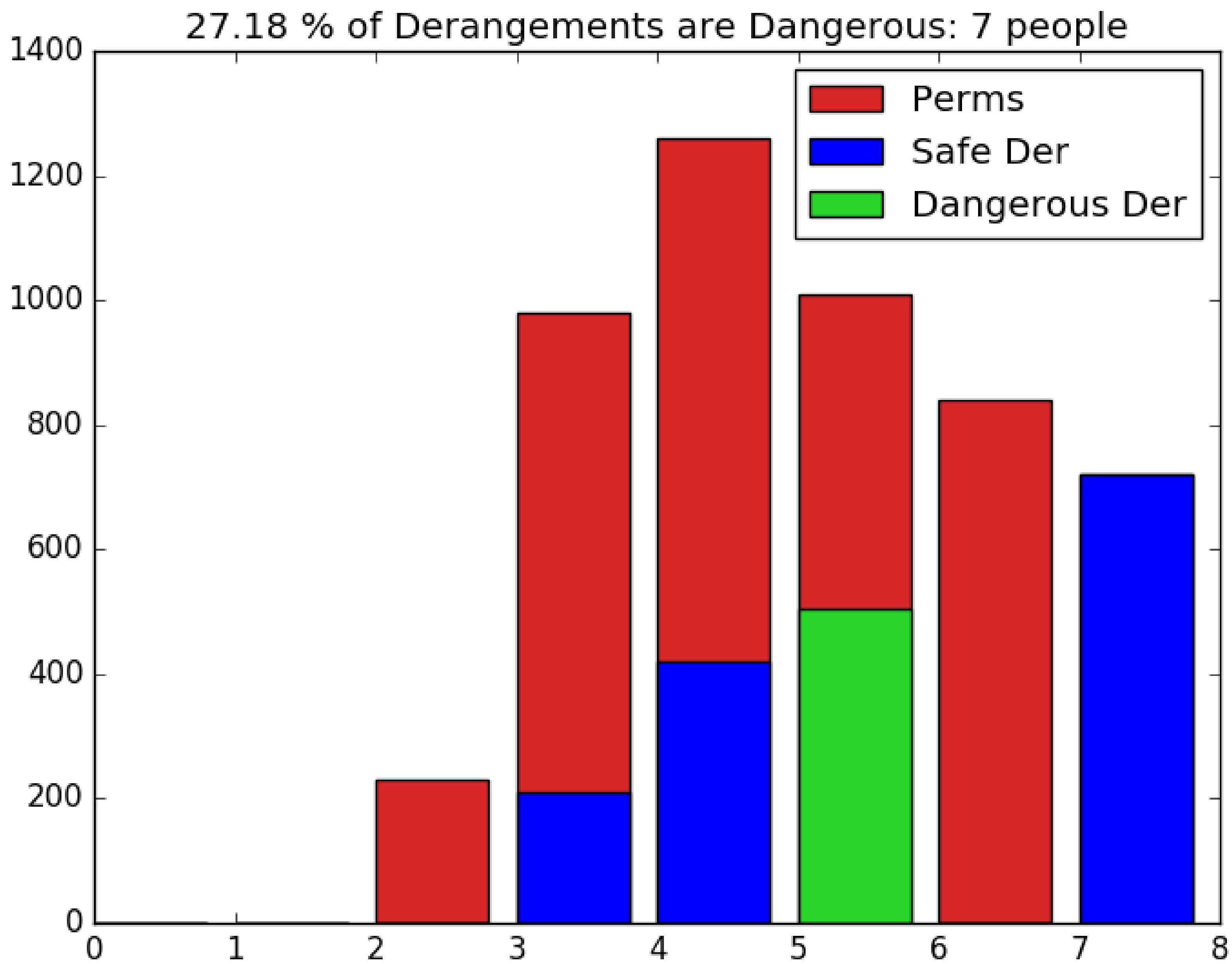


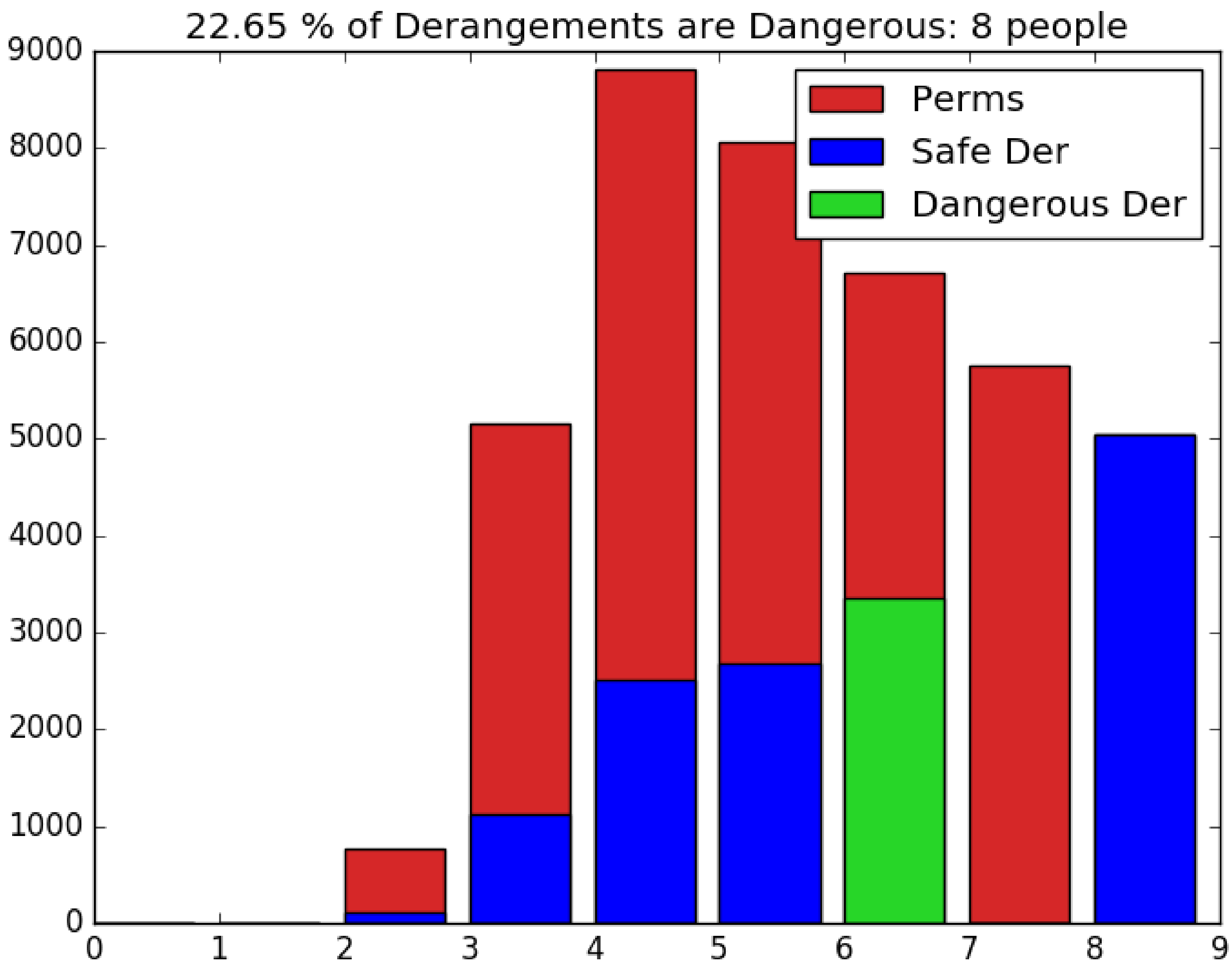


0.0 % of Derangements are Dangerous: 4 people
Perms
Safe Der
Dangerous Der
0
1
2
3
4
5
6
7
8
9

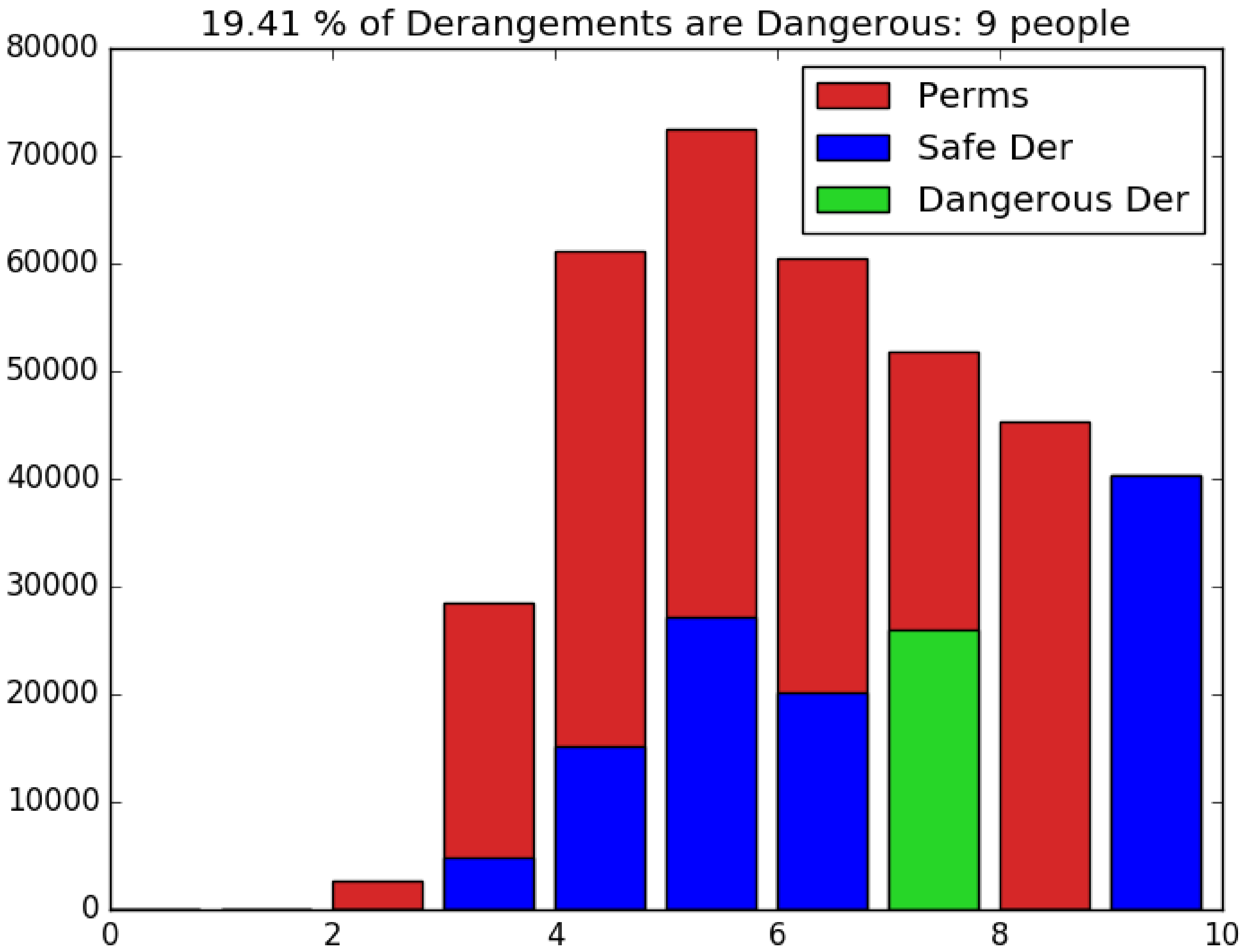

0.0 % of Derangements are Dangerous: 5 people
Perms
Safe Der
Dangerous Der
0
5
10
15
20
25
30
35
40
0
1
2
3
4
5
6

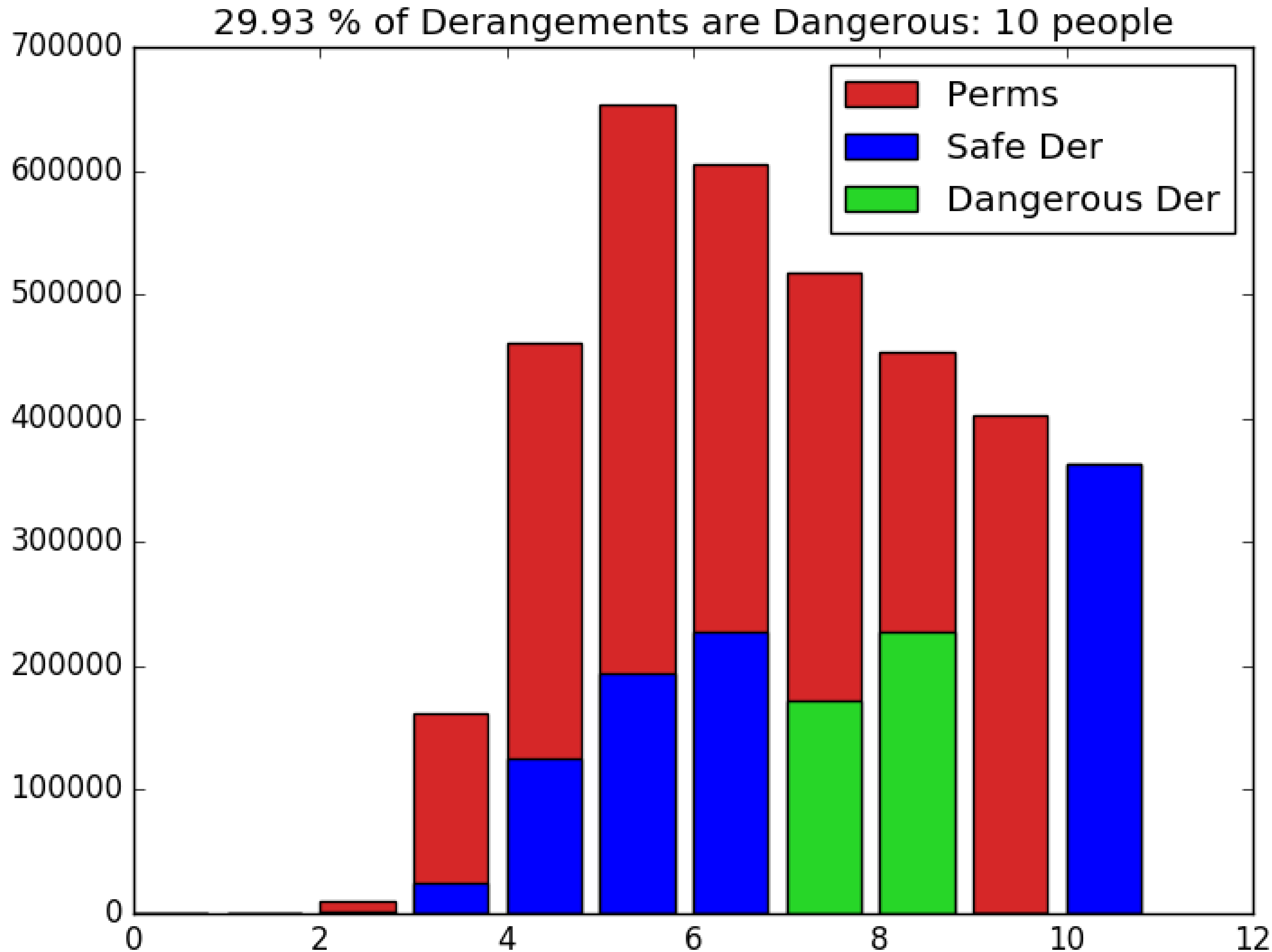

0.0 % of Derangements are Dangerous: 6 people
Perms
Safe Der
Dangerous Der
200
150
100
50
0
0
1
2
3
4
5
6
7

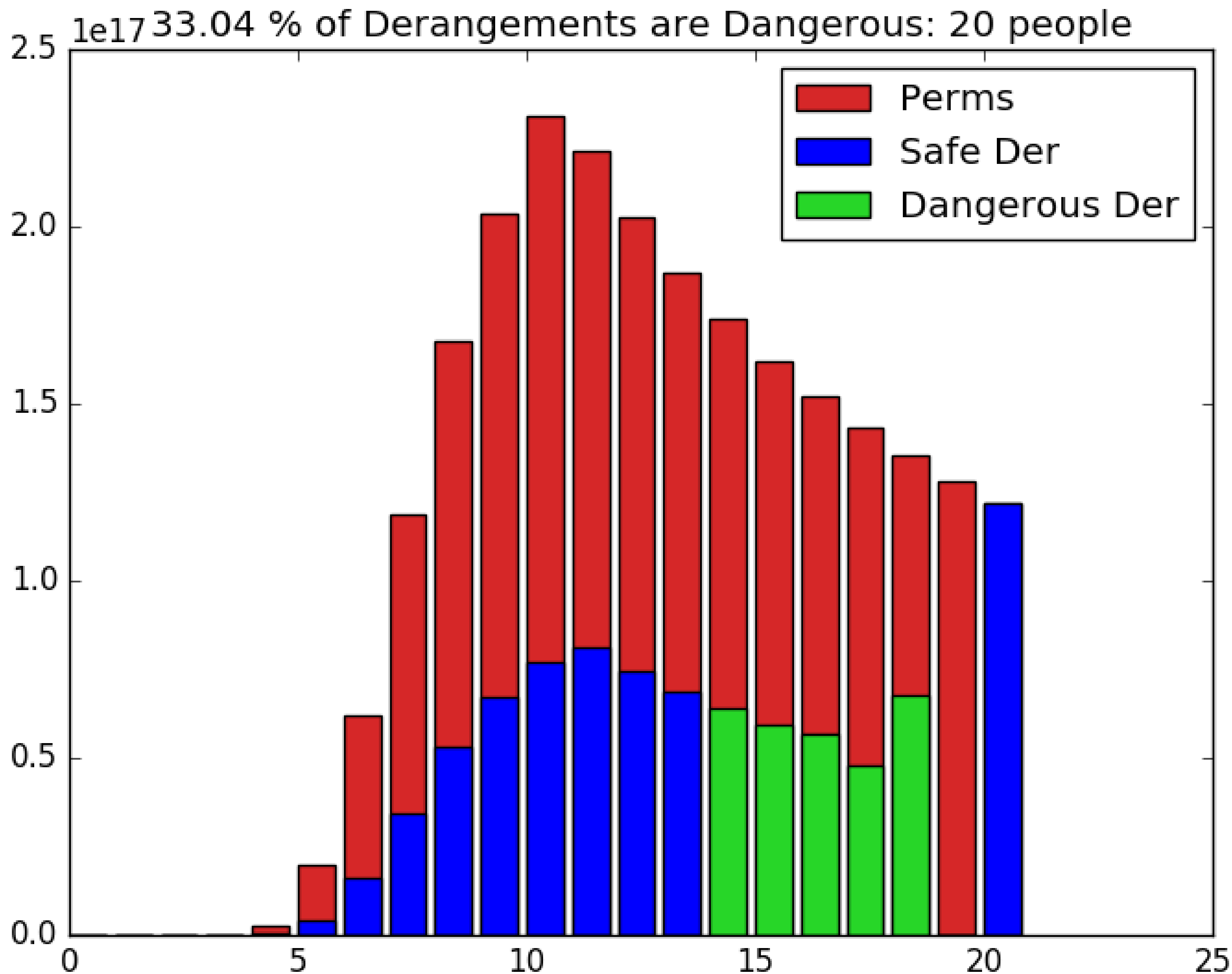

27.18 % of Derangements are Dangerous: 7 people
Perms
Safe Der
Dangerous Der
0
200
400
600
800
1000
1200
1400
0
1
2
3
4
5
6
7
8

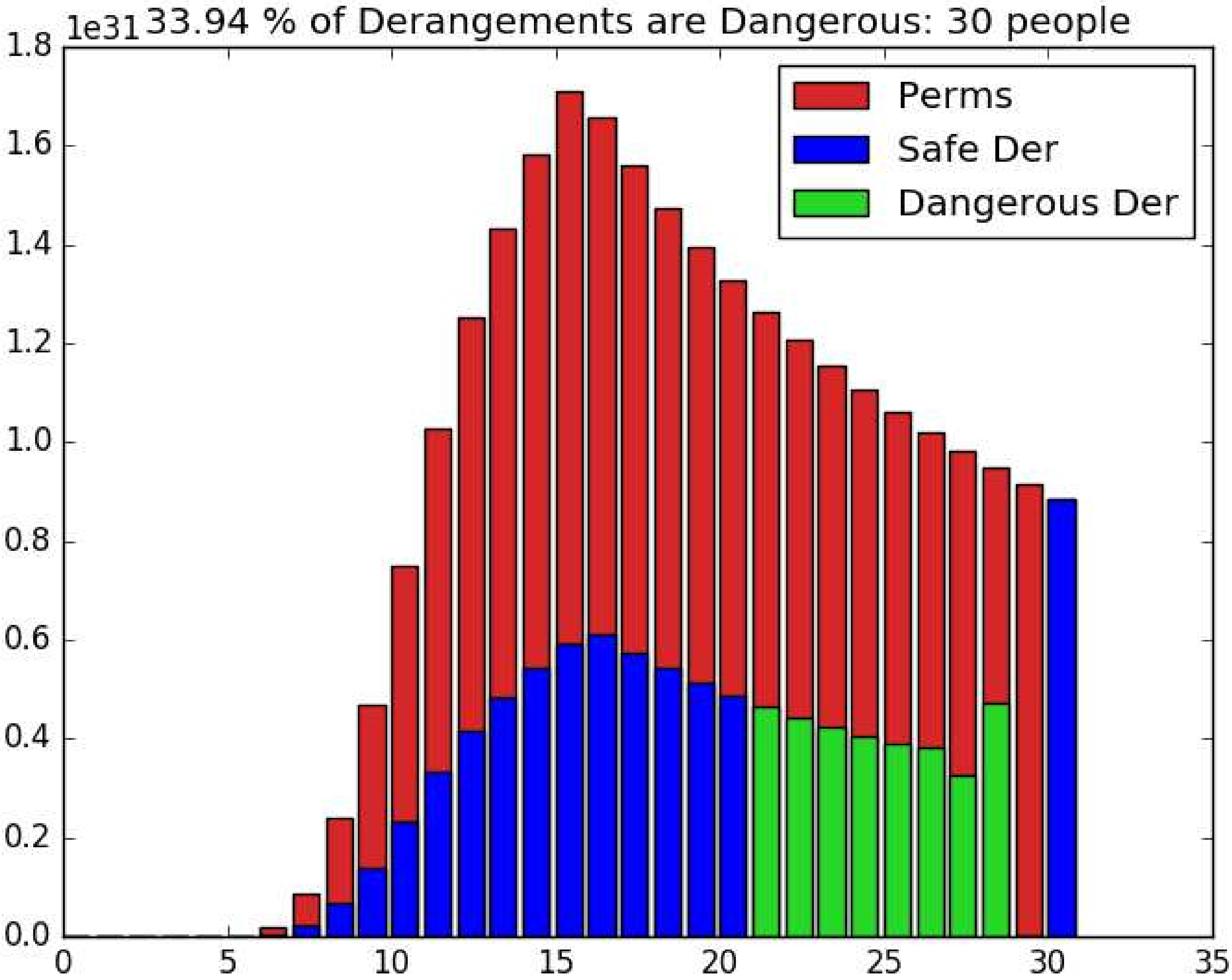


22.65 % of Derangements are Dangerous: 8 people
Perms
Safe Der
Dangerous Der
9000
8000
7000
6000
5000
4000
3000
2000
1000
0
0
1
2
3
4
5
6
7
8
9

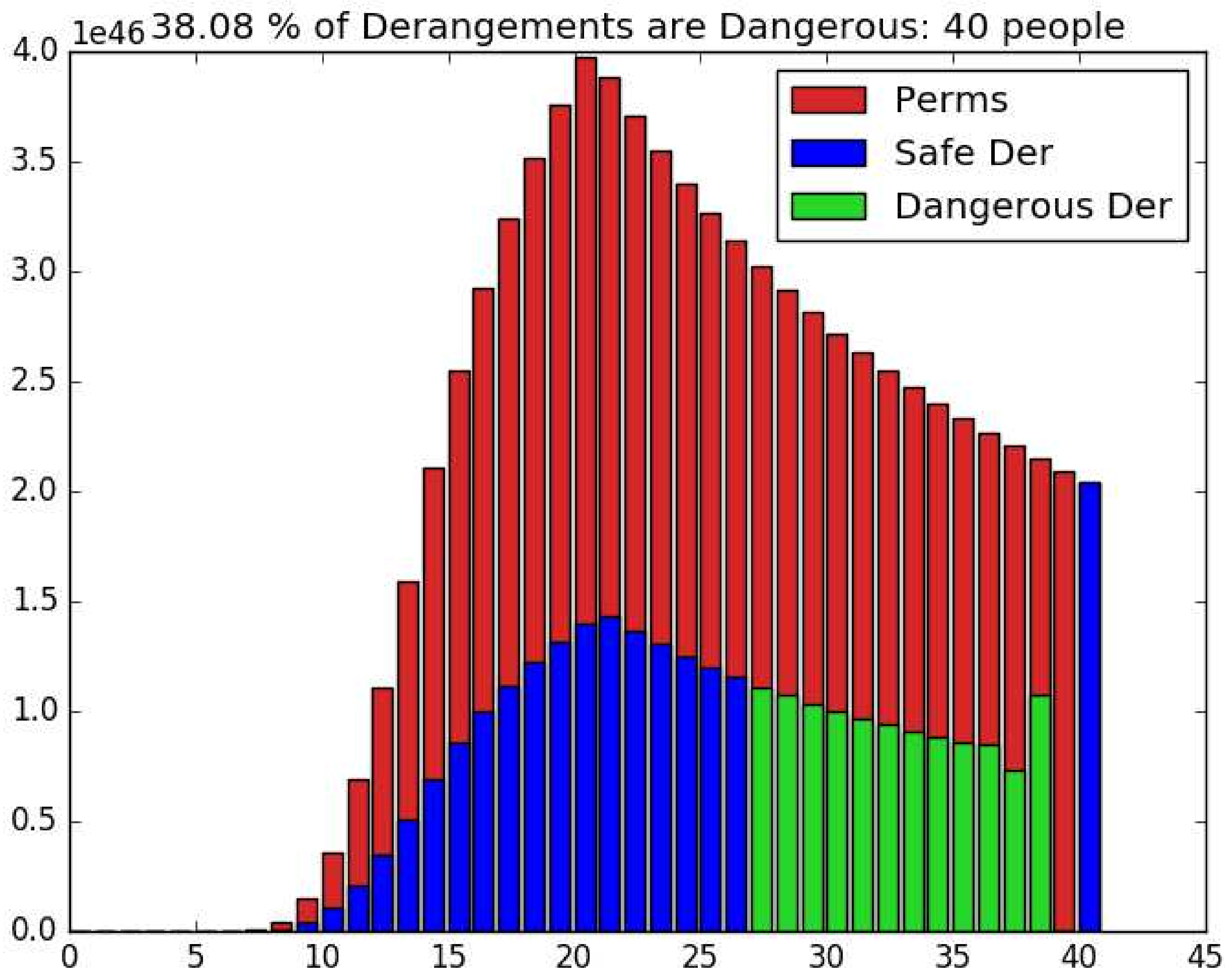

19.41 % of Derangements are Dangerous: 9 people
Perms
Safe Der
Dangerous Der
80000
70000
60000
50000
40000
30000
20000
10000
0
0
2
4
6
8
10


29.93 % of Derangements are Dangerous: 10 people
Perms
Safe Der
Dangerous Der
0
100000
200000
300000
400000
500000
600000
700000
0
2
4
6
8
10
12

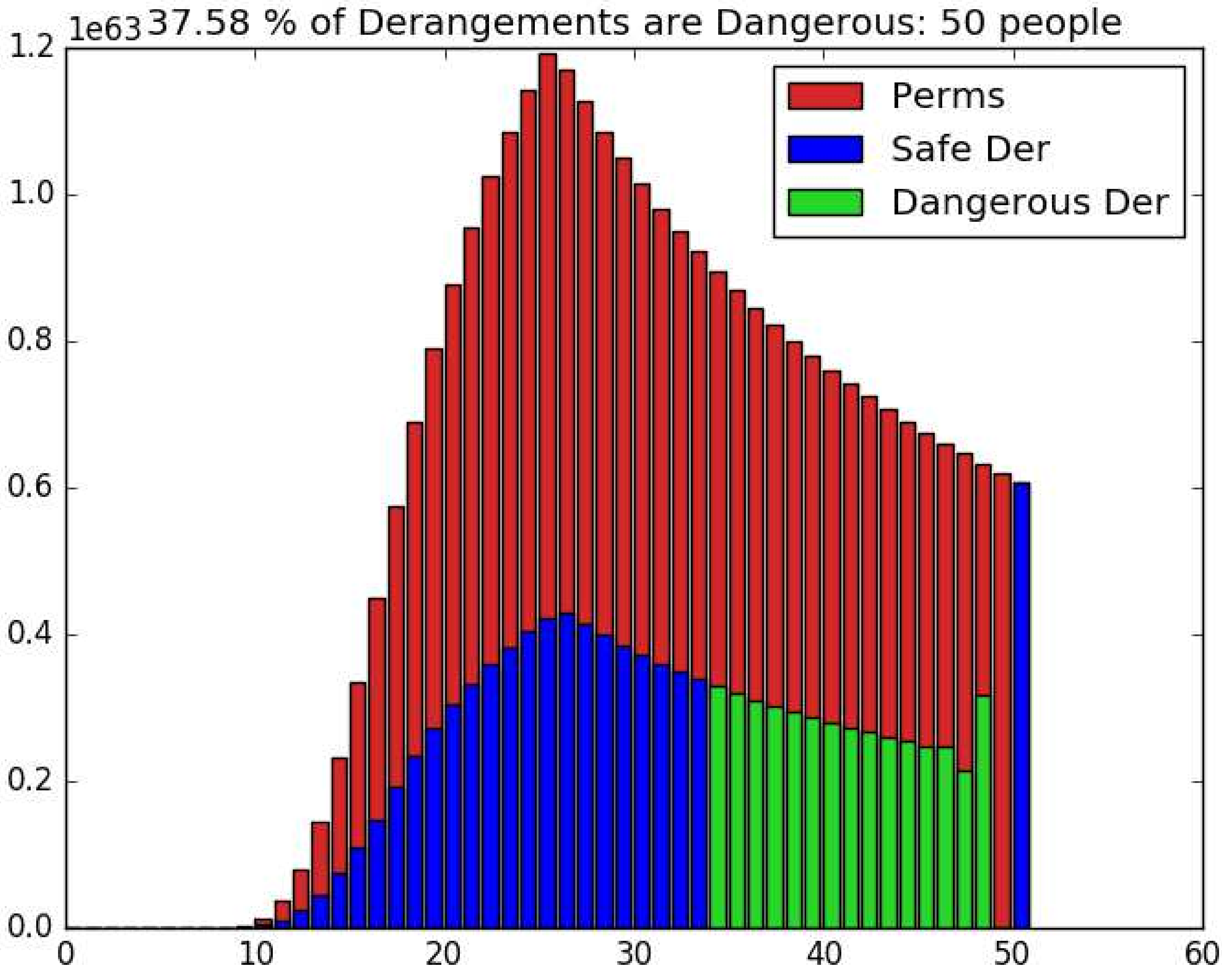

33.04 % of Derangements are Dangerous: 20 people
1e17
2.5
2.0
1.5
1.0
0.5
0.0
0
5
10
15
20
25
Perms
Safe Der
Dangerous Der



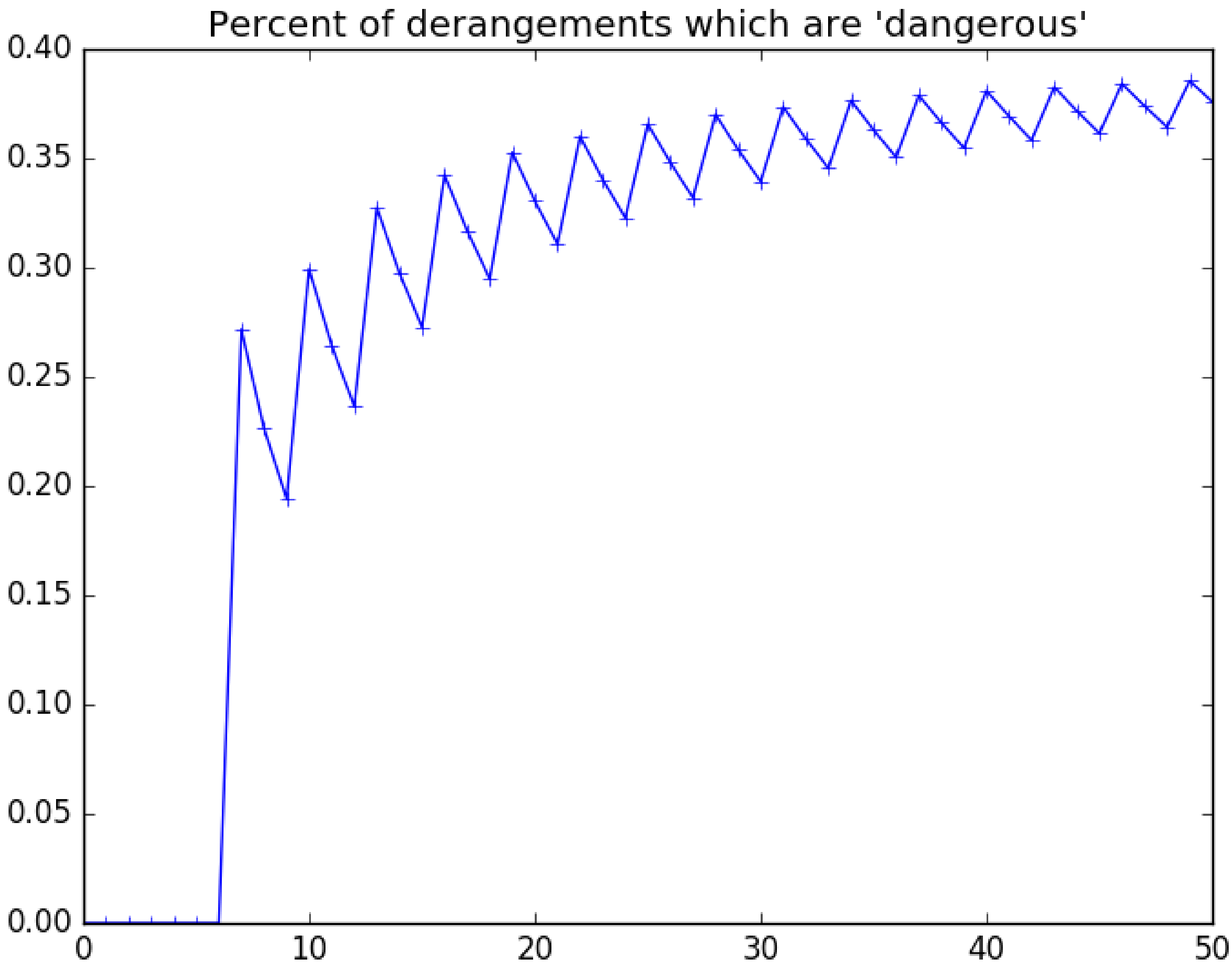
1e31
33.94 % of Derangements are Dangerous: 30 people
Perms
Safe Der
Dangerous Der
1.8
1.6
1.4
1.2
1.0
0.8
0.6
0.4
0.2
0.0
0
5
10
15
20
25
30
35



38.08 % of Derangements are Dangerous: 40 people
1e46
4.0
3.5
3.0
2.5
2.0
1.5
1.0
0.5
0.0
0
5
10
15
20
25
30
35
40
45
Perms
Safe Der
Dangerous Der



37.58 % of Derangements are Dangerous: 50 people
1e63
1.2
1.0
0.8
0.6
0.4
0.2
0.0
0
10
20
30
40
50
60
Perms
Safe Der
Dangerous Der



Percent of derangements which are 'dangerous'
0.40
0.35
0.30
0.25
0.20
0.15
0.10
0.05
0.00
0
10
20
30
40
50

0.385

Third Game:

- What are the dangers?
- We are trying to keep the identities secret until the end of the game, which is the opposite of the previous game where we tried to find the people we were paired with!
- If you bust your cover by putting things directly on your Secret Sister's desk, that's your fault, because lots of people are willing to help you with this step.

NEVER REGIFT!!!

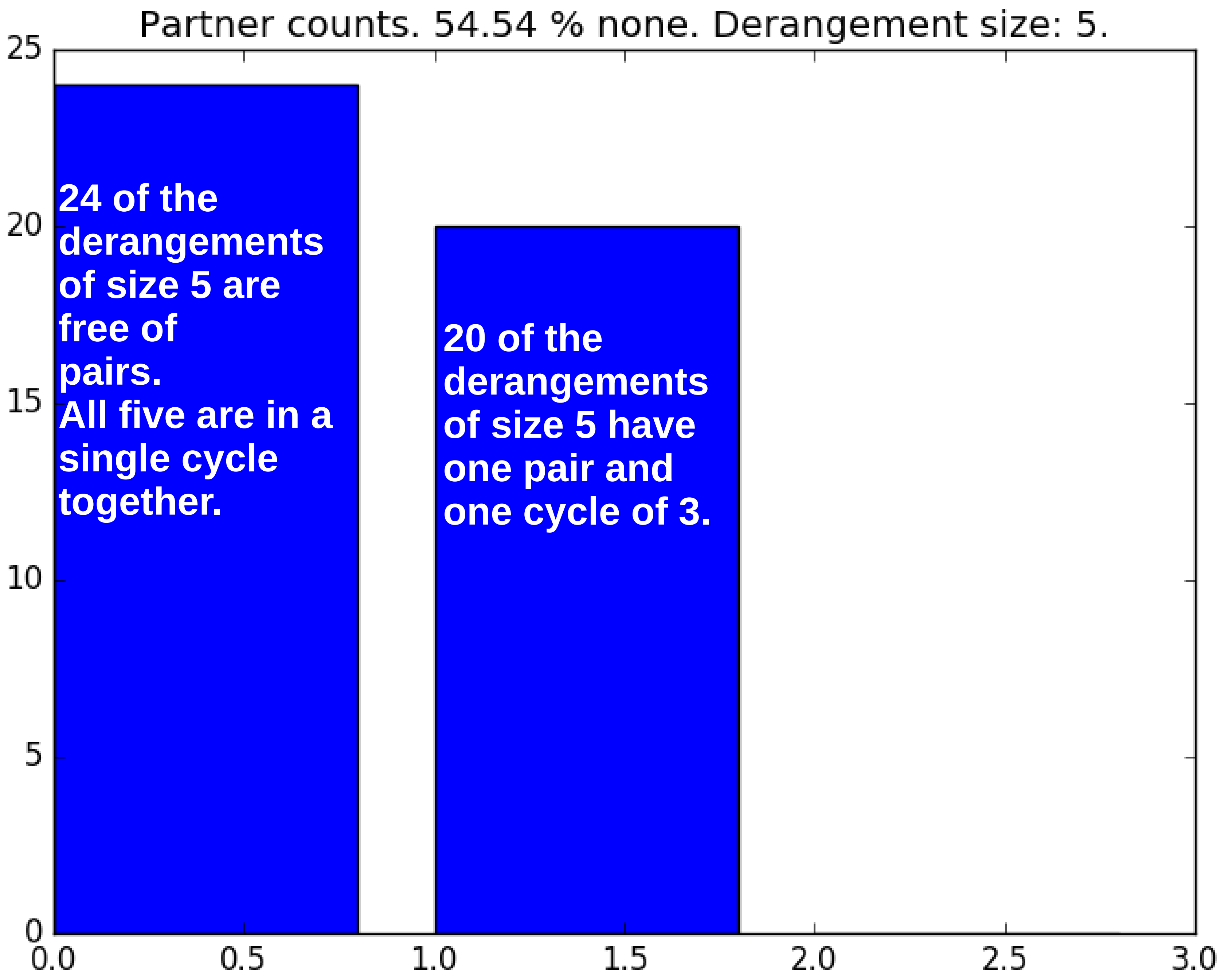
I was in a cycle of size two with Cathy!

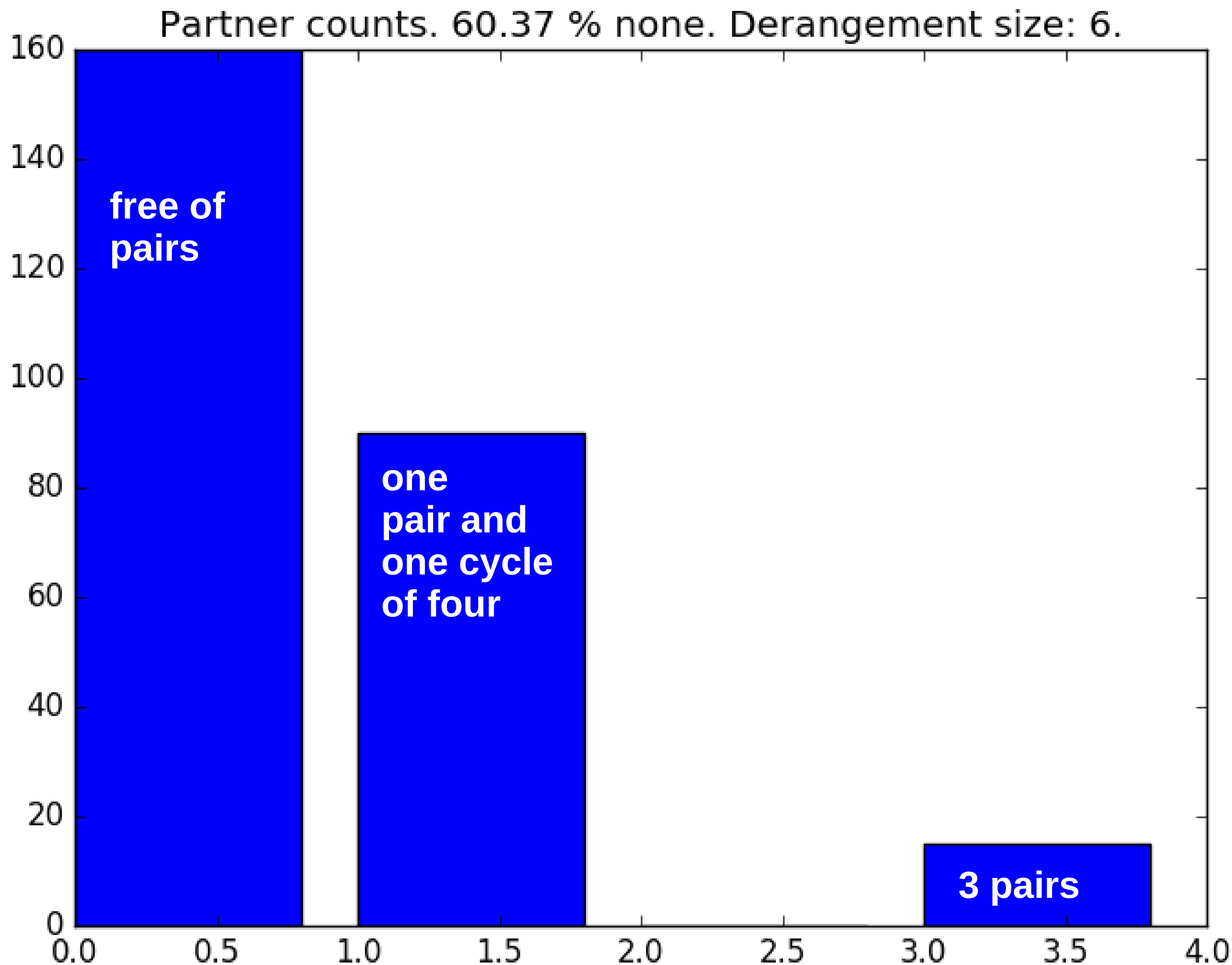
No, I didn't,
but I was tempted to...

What are the chances?

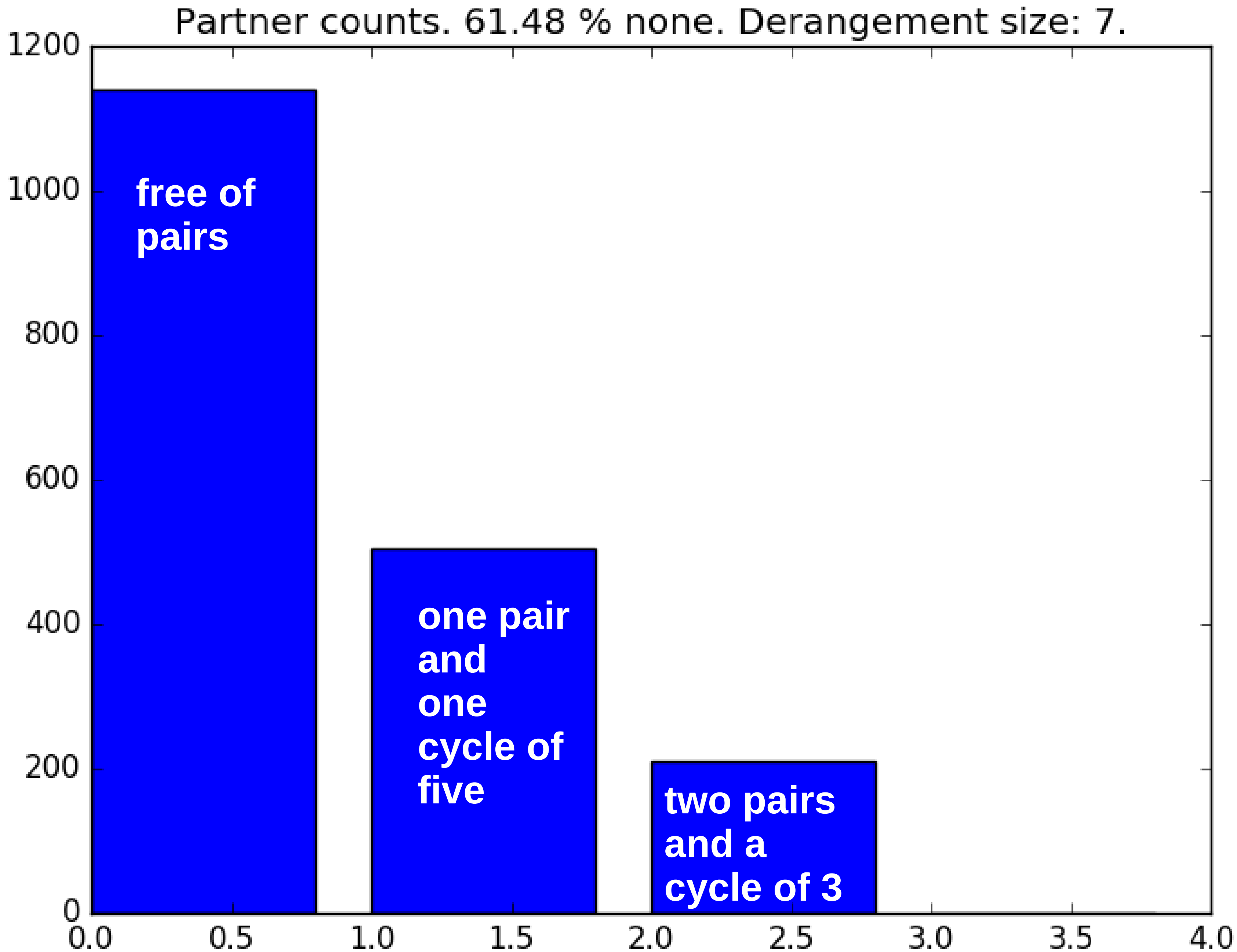
For you personally, at least 1/(n-1).
For the organizer, the expected value is 1.
This is, of course, insane.

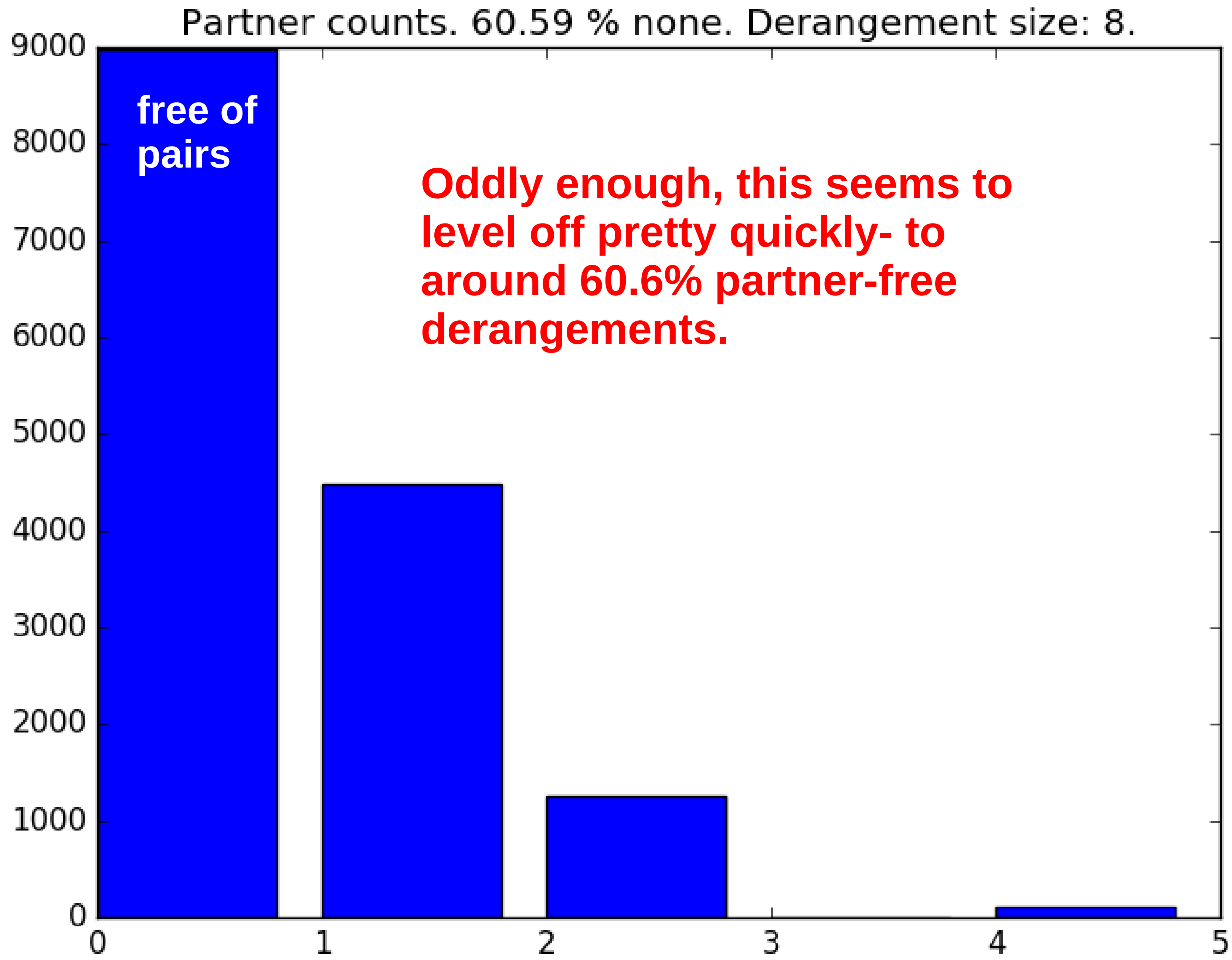
It's like
"The expected value of standard die = 3.5"

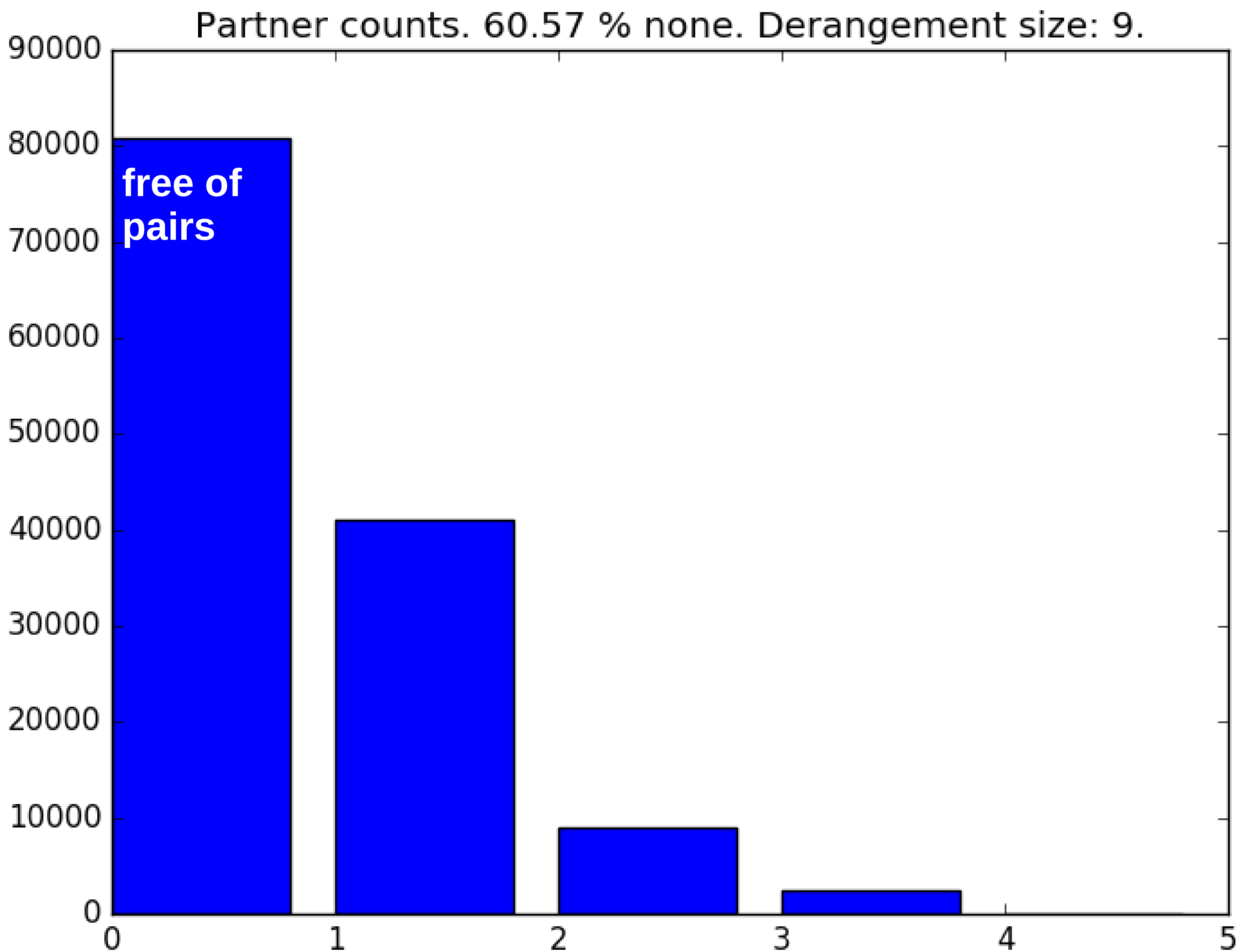

Partner counts. 54.54 % none. Derangement size: 5.
24 of the derangements of size 5 are free of pairs. All five are in a single cycle together.
20 of the derangements of size 5 have one pair and one cycle of 3.
0
5
10
15
20
25
0.0
0.5
1.0
1.5
2.0
2.5
3.0

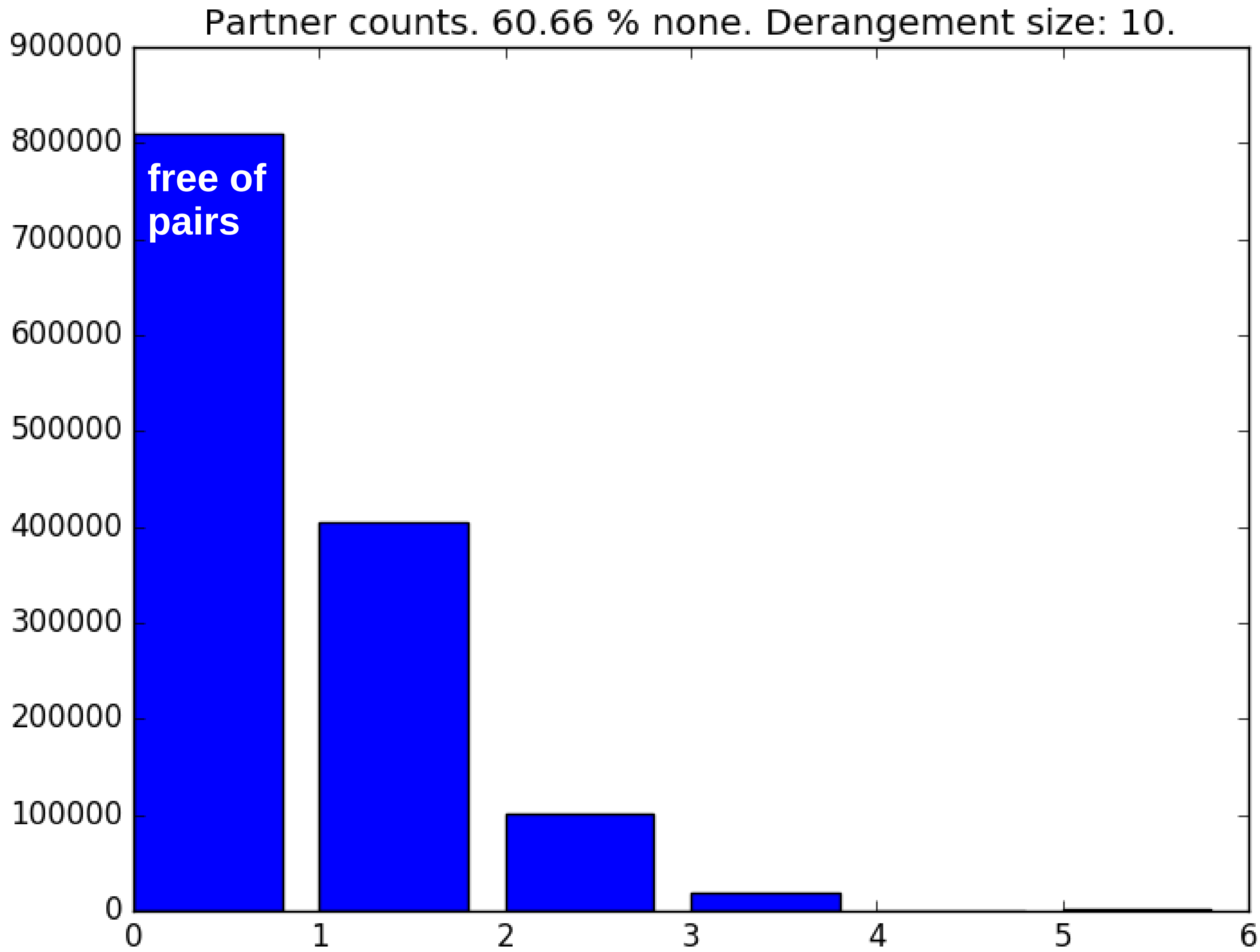


Partner counts. 60.37 % none. Derangement size: 6.
free of pairs
one pair and one cycle of four
3 pairs
0
20
40
60
80
100
120
140
160
0.0
0.5
1.0
1.5
2.0
2.5
3.0
3.5
4.0

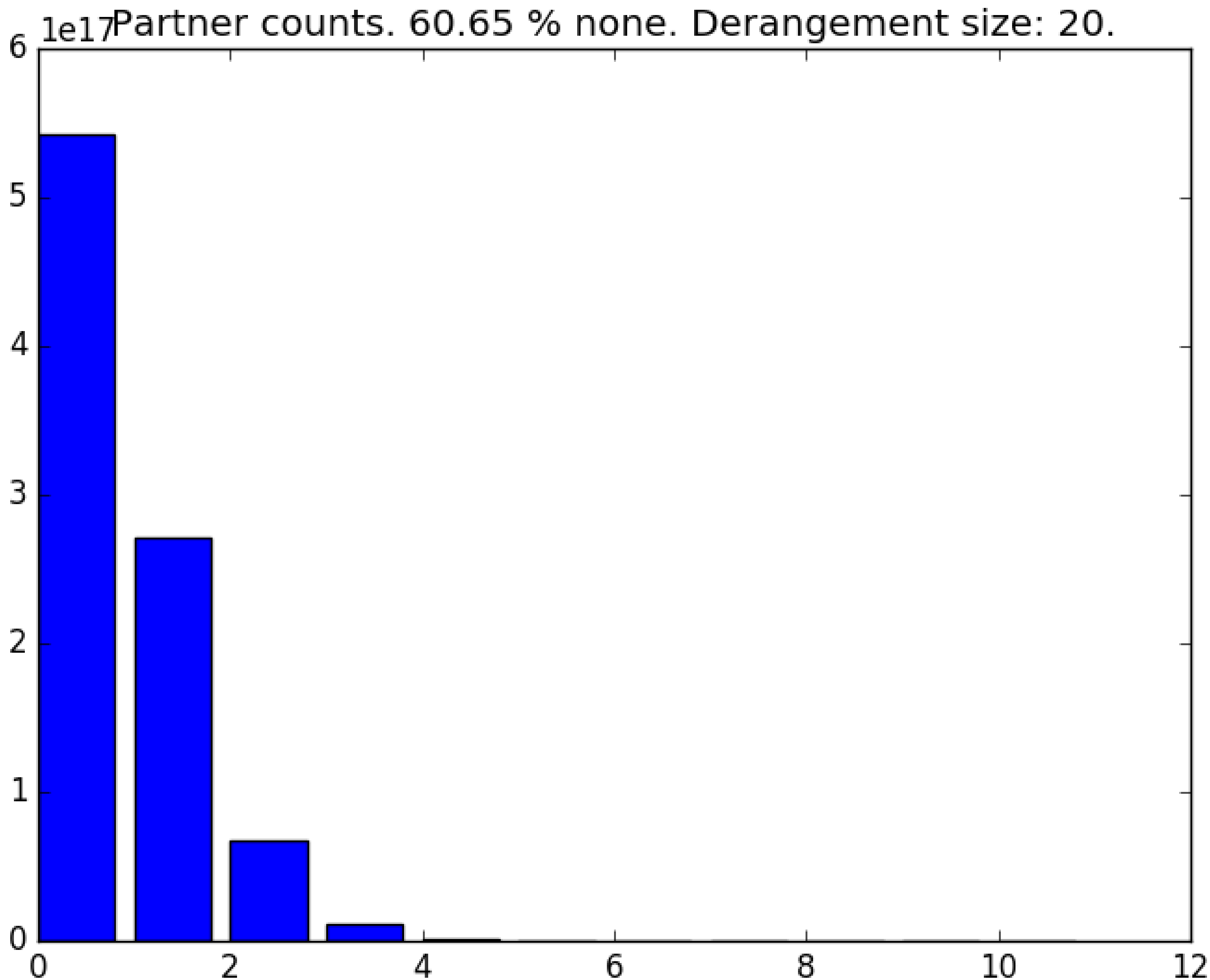

Partner counts. 61.48 % none. Derangement size: 7.
1200
1000
800
600
400
200
0
0.0
0.5
1.0
1.5
2.0
2.5
3.0
3.5
4.0
free of pairs
one pair and one cycle of five
two pairs and a cycle of 3


Partner counts. 60.59 % none. Derangement size: 8.
9000
8000
7000
6000
5000
4000
3000
2000
1000
0
0
1
2
3
4
5
free of
pairs
Oddly enough, this seems to
level off pretty quickly- to
around 60.6% partner-free
derangements.

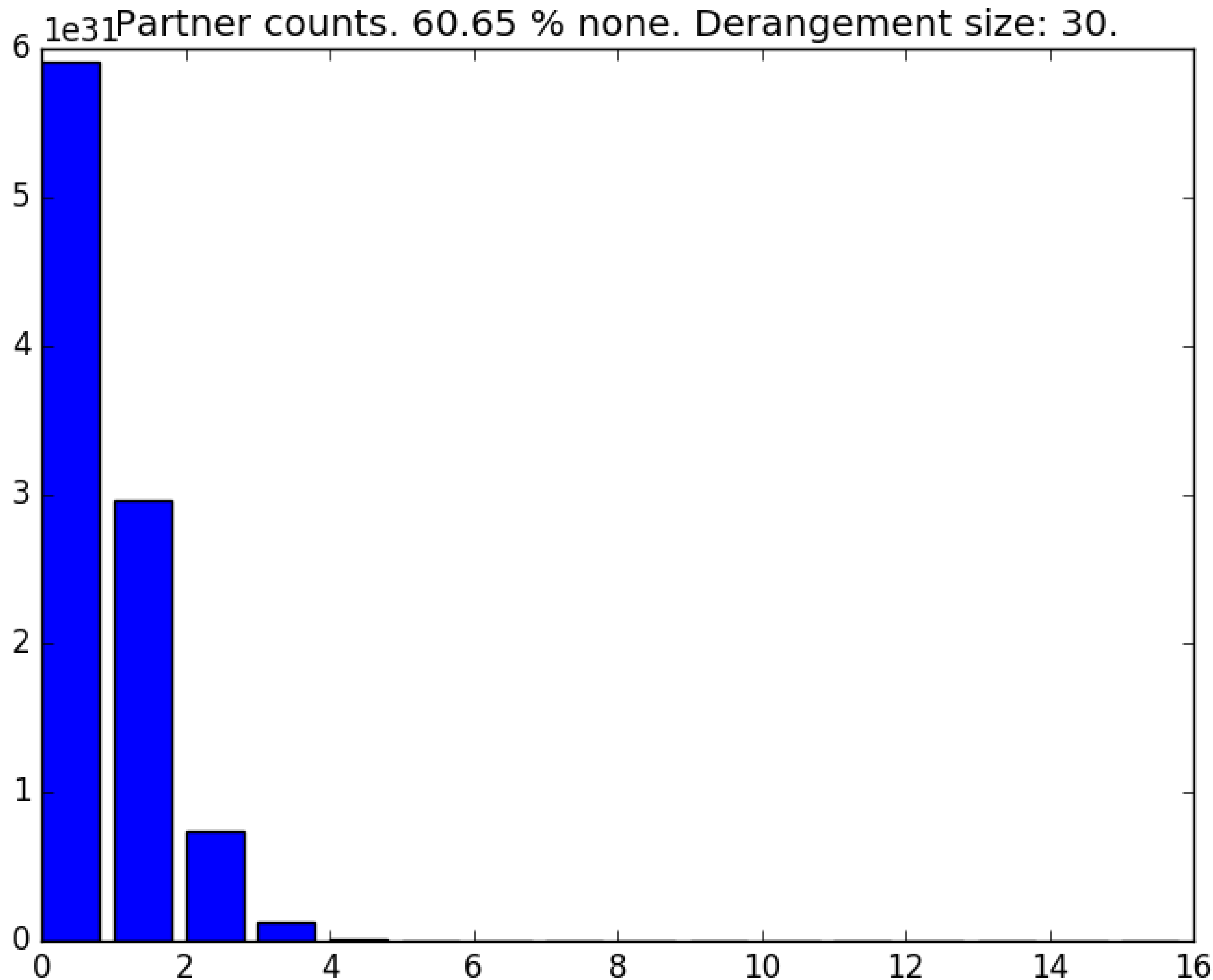

Partner counts. 60.57 % none. Derangement size: 9.
free of
pairs
90000
80000
70000
60000
50000
40000
30000
20000
10000
0
0
1
2
3
4
5

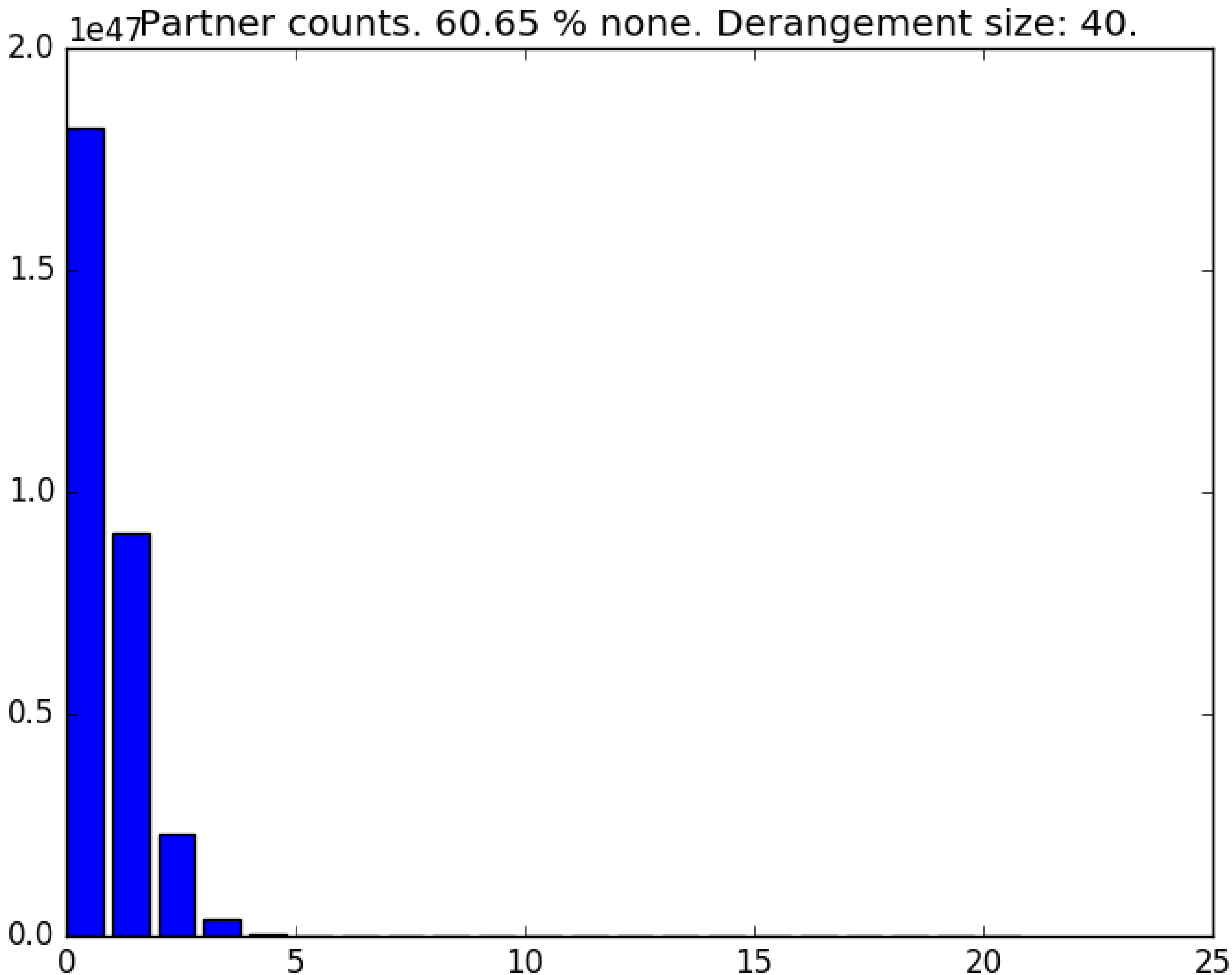


Partner counts. 60.66 % none. Derangement size: 10.
900000
800000
700000
600000
500000
400000
300000
200000
100000
0
0
1
2
3
4
5
6
free of
pairs

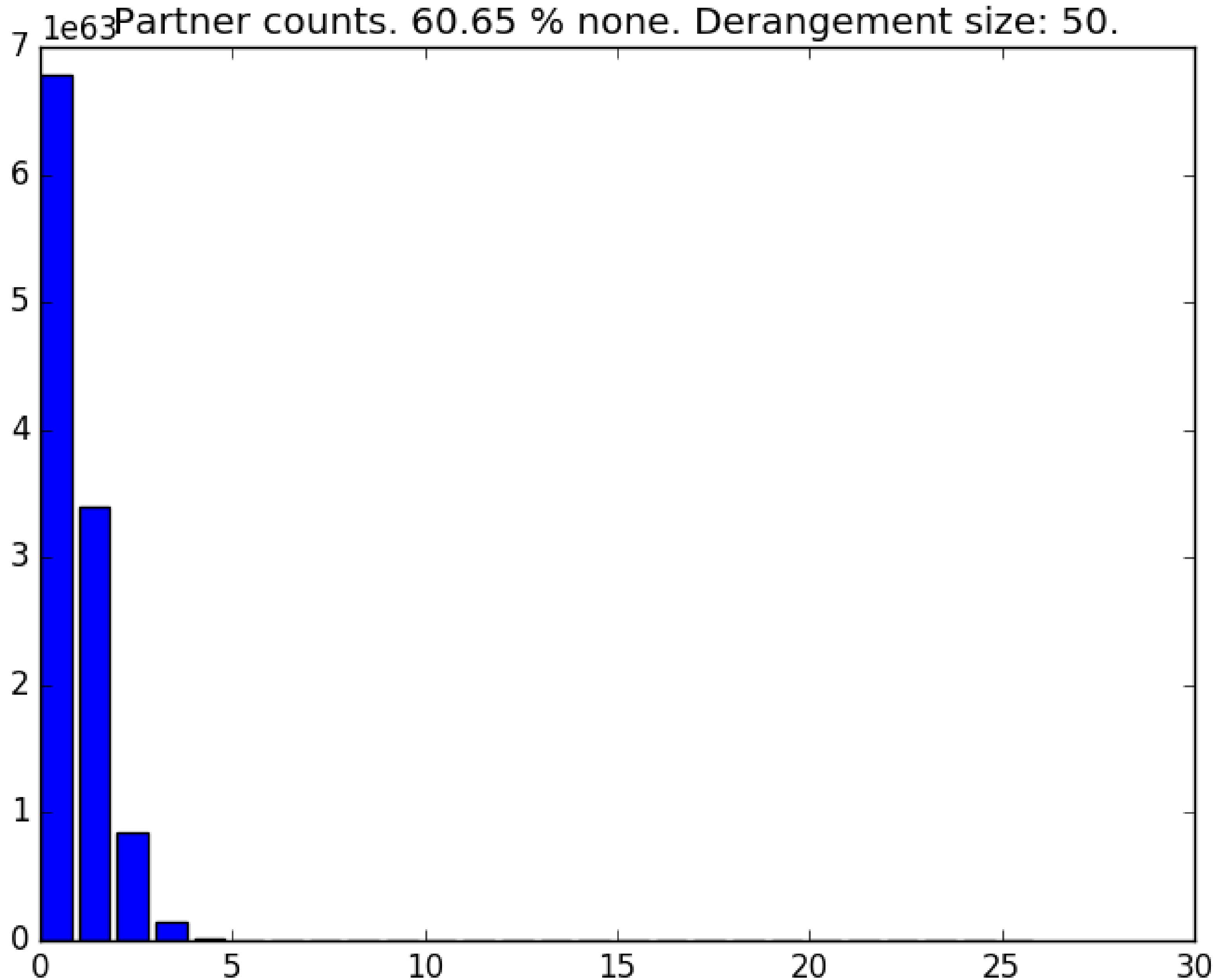


Partner counts. 60.65 % none. Derangement size: 20.
1e17
6
5
4
3
2
1
0
0
2
4
6
8
10
12


Partner counts. 60.65 % none. Derangement size: 30.
1e31
6
5
4
3
2
1
0
0
2
4
6
8
10
12
14
16



Partner counts. 60.65 % none. Derangement size: 40.
1e47
2.0
1.5
1.0
0.5
0.0
0
5
10
15
20
25


Partner counts. 60.65 % none. Derangement size: 50.
1e63
7
6
5
4
3
2
1
0
0
5
10
15
20
25
30

Derangement Games can be socially Dangerous!

YOU HAVE BEEN WARNED!

dietz@american.edu