Geometry: Old and New

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Roanoke College
MAA MD/DC/VA Section Meeting
25 APR 2015

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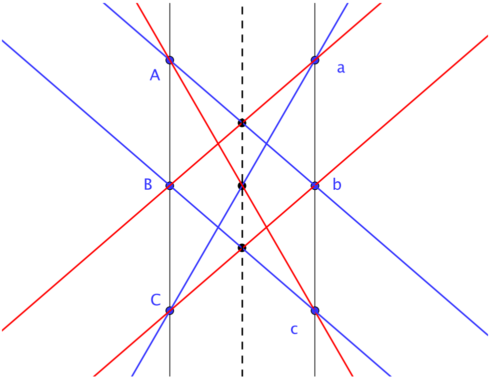
Ancient Geometry

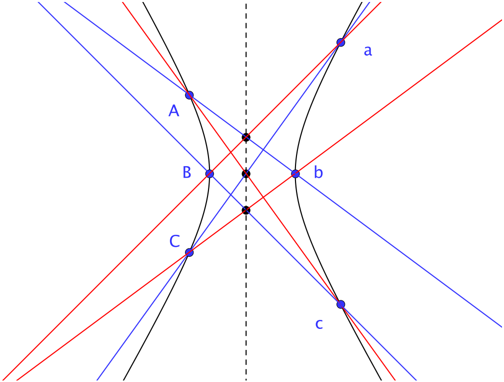
Line Arrangement due to Pappus of Alexandria (Synagogue; c. 340)



Richter-Gebert: 9 proofs in Perspectives on Projective Geometry, 2011

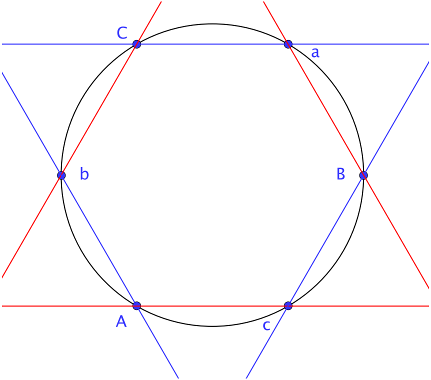
Pascal's Mystic Hexagon Theorem

Pascal: placed the 6 intersection points on a conic (1639)



Braikenridge and MacLaurin: proved the converse

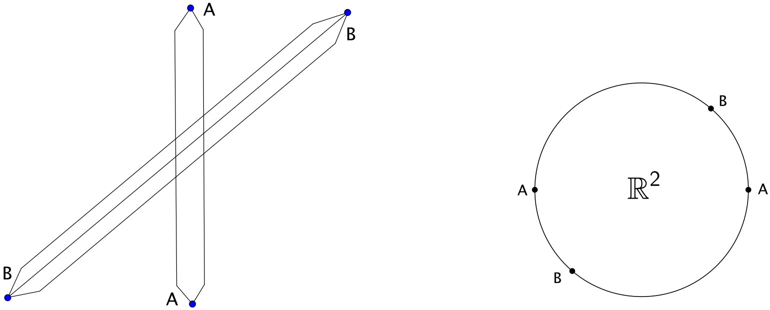
Why Mystic?



The Projective Plane

$\mathbb{P}^2$ is a compactification of $\mathbb{R}^2$

$\mathbb{P}^2 = \mathbb{R}^2 \cup$ line at ∞



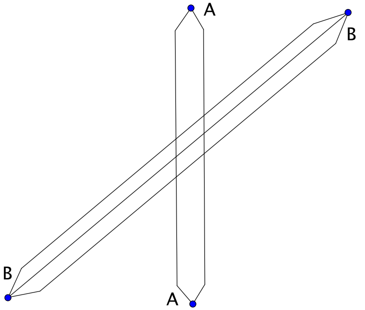
Parallel lines meet at infinity - one point at ∞ for each slope.

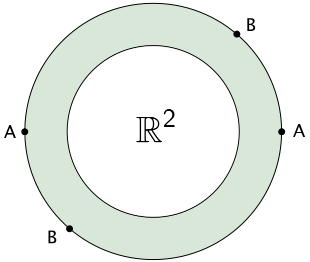
Line at infinity wraps twice around $\mathbb{R}^2$.

The Projective Plane: Thicken Line at Infinity

$\mathbb{P}^2$ is a compactification of $\mathbb{R}^2$

$\mathbb{P}^2 = \mathbb{R}^2 \cup$ line at ∞





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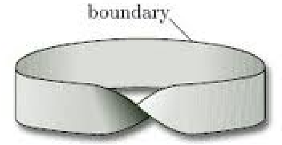
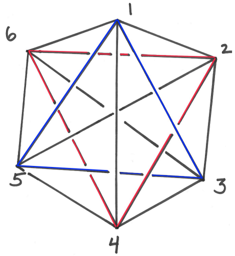
Topology of the Projective Plane

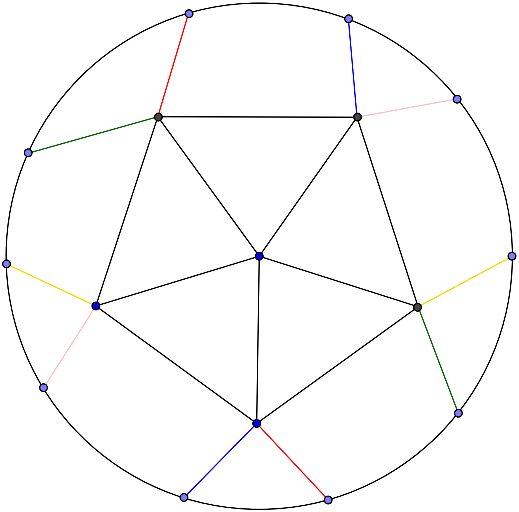
Thicken line at infinity:
$\mathbb{P}^2$ = disk $\cup$ Möbius band



$\mathbb{P}^2$ can't be embedded in $\mathbb{R}^3$
(Conway, Gordon, Sachs (1983):
linked triangles in K_6)



K_6 embedded in $\mathbb{P}^2$



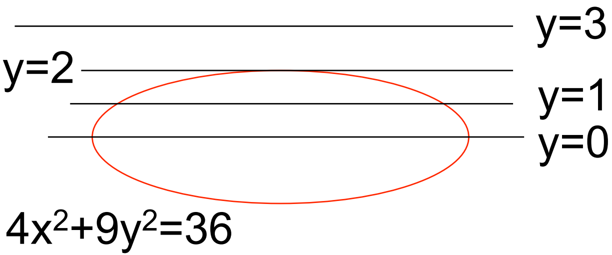
No linked triangles

Bézout's Theorem

Compactness of $\mathbb{P}^2$ allows us to count solutions:

Theorem (Bézout)

Any two curves, without common components, defined by the vanishing of polynomials of degrees d_1 and d_2 meet in $d_1 d_2$ points in $\mathbb{P}^2$, suitably interpreted.

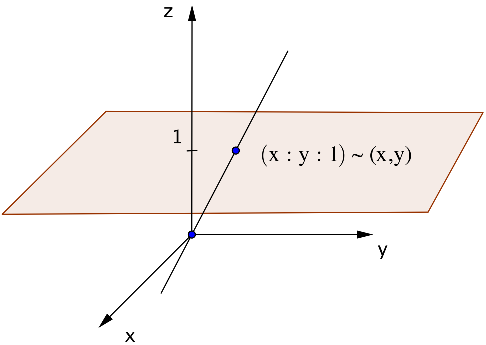


Projective Coordinates: Möbius's model for $\mathbb{P}^2$

Möbius: $\mathbb{P}^2$ = {lines through the origin}
Line through (x, y, z) has coordinates $(x : y : z)$ with

$$(x : y : z) \sim (\lambda x : \lambda y : \lambda z) \text{ for } \lambda \neq 0$$

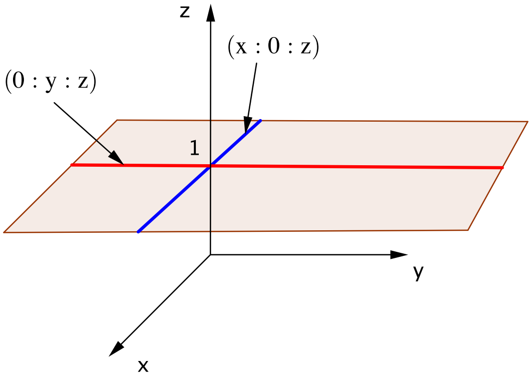
Lines meeting $z = 1$ are of form $(x : y : 1) \sim (x, y)$
Other lines (in xy-plane) form points at infinity $(x : y : 0)$



Lines in $\mathbb{P}^2$: I

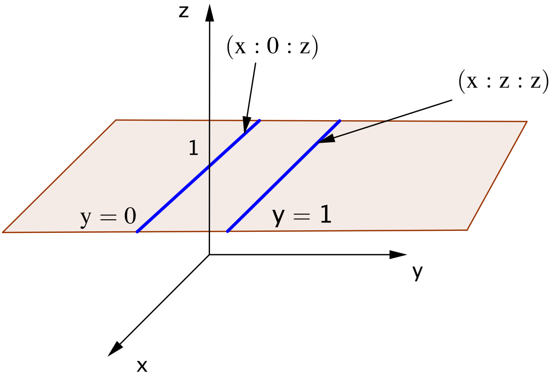
Lines in $\mathbb{P}^2$ correspond to planes through origin

Line $\{(x : 0 : z)\}$ meets line $\{(0 : y : z)\}$ at point (0:0:1).



Lines in $\mathbb{P}^2$: II

Parallel lines: $\{(x : 0 : z)\}$ meets $\{(x : z : z)\}$ at point (1:0:0).



Homogenization

Can't talk about the parabola $y = x^2$:

If we scale the coordinates $(x : y : z) = (\lambda x : \lambda y : \lambda z)$ then we'd require

$$\lambda y = \lambda^2 x^2$$

for all λ (true only for the origin $(x, y) = (0, 0)$).

Curves in $\mathbb{P}^2$ are defined by the vanishing of **homogeneous** polynomials:

$$y - x^2 = 0 \longleftrightarrow yz - x^2 = 0$$

Bashelor's Work

Enumerative Geometry of Conics: How many conics pass through p points and are tangent to ℓ lines and c conics in general position?

Conics through 5 points

$$ax^2 + bxy + cy^2 + dx + ey + f = 0$$
$$ax^2 + bxy + cy^2 + dxz + eyz + fz^2 = 0$$
$$(a : b : c : d : e : f) \in \mathbb{R}^6 / \sim = \mathbb{P}^5$$

Point conditions force $(a : b : c : d : e : f)$ to lie on a hyperplane.

$$(x_0 : y_0 : z_0) \text{ on } C \iff ax_0^2 + bx_0y_0 + cy_0^2 + dx_0z_0 + ey_0z_0 + fz_0^2 = 0.$$

Intersecting 5 hyperplanes in $\mathbb{P}^5$ gives a single point corresponding to the one conic through all 5 points.

Determinants

Given three points in $\mathbb{P}^2$ we can list them as columns of a 3x3 matrix,

$$\left[\begin{array}{c|c|c} a_x & b_x & c_x \\ a_y & b_y & c_y \\ a_z & b_z & c_z \end{array}\right].$$

The determinant $[abc]$ of this matrix measures six times the volume of the tetrahedron with edges a, b and c.

$[abc] = 0 \iff$ points a, b, c are collinear.

Can recover coordinates from full knowledge of determinants.

Plücker Relations

The determinants of the 3x3 submatrices of a larger matrix satisfy quadratic relations.

For example, given five points in $\mathbb{P}^2$ we form the matrix

$$\left[\begin{array}{c|c|c|c|c} a_x & b_x & c_x & d_x & e_x \\ a_y & b_y & c_y & d_y & e_y \\ a_z & b_z & c_z & d_z & e_z \end{array}\right].$$

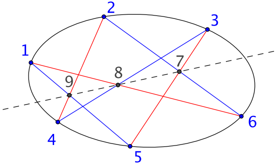
Cramer's Theorem implies that

$$[abc][ade] - [abd][ace] + [abe][acd] = 0.$$

In particular, if $[abc] = 0$ then $[abe][acd] = [abd][ace]$.

Conics and Pascal's Theorem

Six points lie on a conic
$\iff$ $[abc][aef][dbf][dec]$
$= [def][dbc][aec][abf]$.



$$
\begin{array}{lcrcl}
\textbf{conic:} & \Longrightarrow & [125][136][246][345] & = & +[126][135][245][346] \\
[159]=0 & \Longrightarrow & [157][259] & = & -[125][597] \\
[168]=0 & \Longrightarrow & [126][368] & = & +[136][268] \\
[249]=0 & \Longrightarrow & [245][297] & = & -[247][259] \\
[267]=0 & \Longrightarrow & [247][268] & = & -[246][287] \\
[348]=0 & \Longrightarrow & [346][358] & = & +[345][368] \\
[357]=0 & \Longrightarrow & [135][587] & = & -[157][358] \\
\hline
[257][987]=0 & \Longleftarrow & [297][587] & = & +[287][597]
\end{array}
$$

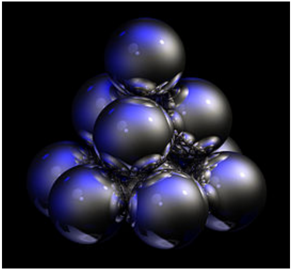
If points 2, 5, and 7 are not collinear, $[257] \neq 0$ so $[987] = 0$.

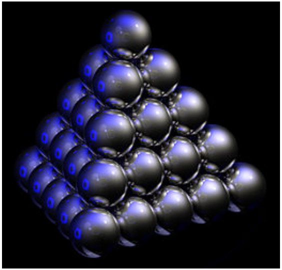
Computer Assistance I: Conjecture + Proof

Conjecture (Kepler)

No packing of spheres covers more than $\pi/3\sqrt{2}$ (approximately 74%) of the filled volume.

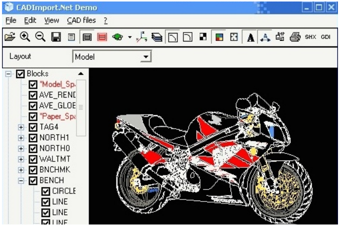
History: Harriot and Sir Walter Raleigh (1591) and Kepler (1611)
Gauss (regular lattices; 1631) and Fejes Tóth (1953)
Tom Hales and Sam Ferguson (1992-2006)
FlysPecK (Formal Proof of Kepler; 2014)





Computer Assistance II

Prediction: Computers will become our mathematical assistants, vastly raising the level of our mathematical reasoning (c.f. advanced chess and CAD/origami).



Computer Assistance III

We already have strong computer and robotic assistance:

Mind controlled Deka-arm

Wild Conjecture: In the (perhaps distant) future the division between human and computer will become less and less distinct.

Time Magazine 2011

My current work

There is at least one degree 3 curve through every set of 9 points.

Question

When do 10 points lie on a plane curve of degree 3?

Smooth curves of degree 3 are called elliptic curves and play a role in both **elliptic curve cryptography** and in Wiles's proof of **Fermat's Last Theorem**.

The 10 points lie on a cubic when the determinant of a 10×10 matrix is zero (25 million terms).

Reiss (1842) wrote out a 20 term polynomial of degree 10 in the 3×3 brackets that computes the determinant faster.

Ruler and Compass Construction

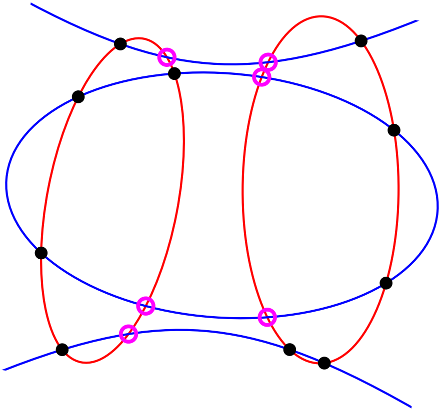
Construction (T and Wehlau)

We developed a straightedge-and-compass construction that checks whether 10 points lie on a cubic.

Such constructive questions are now back in vogue since they leverage all sorts of ideas in computational geometry.

The Key Idea: Cayley-Bacharach

10 points on a cubic precisely when 6 auxiliary points on a conic.



We construct the 6 points using straightedge and compass and then invoke Pascal's Theorem.

Straightedge Construction

Meet and Join Algebra: Allows algebraic formulation of straightedge-only constructions

Theorem (after Sturmfels and Whiteley)

There exists a straightedge construction to determine if 10 points lie on a cubic (with about 100 million lines).