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A post BC calculus inquiry enrichment course

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- I attended the IBL workshop at UCSB early last summer – wanted to try something out
- I have always had IBL-ish tendencies
- Have taught honors and regular calculus repeatedly, also history of math and I run a math circle so have lots of interesting material, with no syllabus to cover for this course

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- They wanted more topics sampled rather than a shorter, deeper list – their choice.

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- $y = g(f(x))$ is built using an intermediate result $u = f(x)$, $y = g(u)$:

  $du = f'(x)\,dx,\; dy = g'(u)\,du,$ so by **substitution** $dy = g'(u)f'(x)\,dx$

  This says that the derivatives get multiplied when differentiating a composite function, just as slopes did in the purely linear case.

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and similarly

$$g_1(u,b) = \begin{cases} \frac{g(u)-g(b)}{u-b} & \text{if } u \neq b; \\ g'(b) & \text{if } u = b \end{cases}$$

Chain Rule Proof – DAY 1 – continued

Notice that we can then write the following equations for the differences:

$$g(u) - g(b) = g_1(u, b)(u - b),\ f(x) - f(a) = f_1(x, a)(x - a).$$

This allows us to “divide and then conquer” the composition and extends the differentials language to finite differences:

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$$\begin{aligned} g(f(x)) - g(f(a)) &= g(u) - g(b) = g_1(u, b)(u - b) \\ &= g_1(u, b)(f(x) - f(a)) \\ &= g_1(u, b) f_1(x, a)(x - a) \end{aligned}$$

which permits the division of the composite by $x - a$ and the passage to the limit:

Chain Rule Proof – DAY 1 – continued

$$\lim_{x \to a} \frac{g(f(x)) - g(f(a))}{x - a} = \lim_{x \to a} g_1(u, b) f_1(x, a) = g'(b) f'(a)$$

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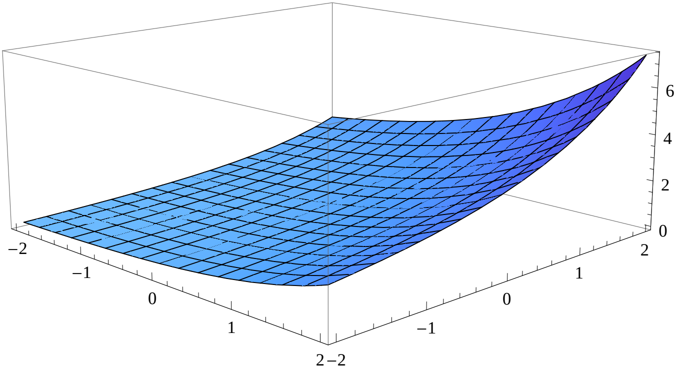
On computer, I zoomed in to show why we care about tangent lines, then showed some plots of difference quotients, with both variables x and a as well as with x alone and $h = x - a$ as button

Difference quotient plots

Here is a plot of the difference quotient for the exponential function with *x* and *a* on $[-2, 2]$

Difference quotient plots

Here is a plot of the difference quotient for the exponential function with x and a on $[-2, 2]$



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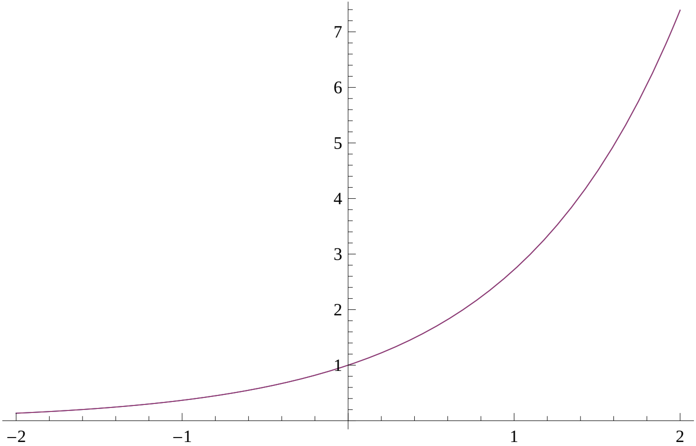

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In notion of best practices: took to heart the importance of day 1 in setting tone and expectations (UCSB workshop advice)

Some General Comments

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- History

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- Fun history here: d'Alembert vs. Gauss

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- Back to divisibility of p(z)-p(a) by z-a for n roots! Also back to intermediate value theorem, topological degree for real polynomials

My other favorites – hard to choose

- Numerical integration (ideas behind error estimates) – use Taylor polynomials; use trapezoid for Stirling's (below) and sum of powers of integers.
- Theory and Proof: Newton on integrability in monotonic case; discussion on need for concept of uniform continuity (or something like it) for general continuous functions.
- History: Fermat and geometric partitions; Archimedes; Tangents: Fermat vs. Descartes; Newton estimating π; Euler summing $\zeta(2)$
- Fixed point iteration: Babylonian $\sqrt{2}$; geometric series; general fixed point and rate of convergence; Newton's method.
- Series as sums of differences – Cauchy criterion coming naturally;
- Size of n! tied to Taylor series; radius of convergence and complex values; Abel's theorem on convergence at endpoints.

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Thank you for your attention and I welcome your comments and/or questions.

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- Student who asked for this was really happy we did it!!

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- Subtract to show exponential (geometric) approach to limiting value (don't find partial sum directly):
$$s_n - S = r(s_{n-1} - S)$$

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Cauchy Criterion for Series

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- No more oops. If limit is to exist, for any k the Cauchy tail must tend to zero.

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- Always a rectangle, always with rational sides! Limit is irrational.

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- Get everything good except $\sqrt{2\pi}$ (get e instead)

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