Ye Olde Fundamental Theorem of Algebra

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April 14, 2012

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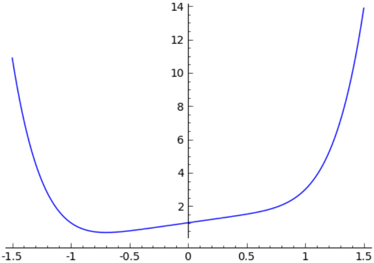
I will discuss an old method for factoring, which was generalized to give one of the more-algebraic proofs of the FTA.

More on $x^6 + x + 1$

Here is a graph of $y = x^6 + x + 1$:



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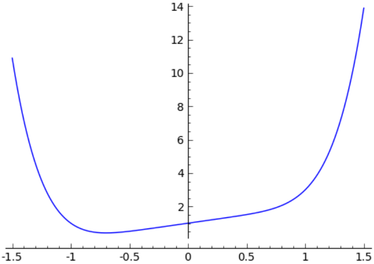

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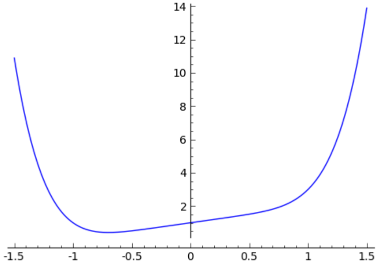

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We need a quadratic factor.

Descartes/Hudde method

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- Hudde's letter contains 22 rules for "reducing" equations in one variable—essentially for factoring polynomials.

Descartes/Hudde method: XIX Regula

Sit æquatio Propoſita, ut ante,

$$x^6 * + qx^4 + rx^3 + sxx + tx + v \infty 0,$$

& inquiratur num dividi poſſit per æquationem duarum dimenſionum cui nullus terminus deſit, pone per $xx + yx + w \infty 0$. ſi itaque per eam diviſibilis ſit, erit $xx \infty -yx - w$, quo valore ipſius xx, ubique in locum xx ſubrogato, reſultabit æquatio in qua x unam tantùm habebit dimenſionem, nimirum

$$\begin{array}{ll} -3wwyx & -w^3 \quad \infty 0. \\ -y^5 & -wy^4 \\ +4wy^3 & +3wwyy \\ -qy^3 & +qww \\ +2qwy & +rwy \\ -rw & -sw \\ +yyr & -qwyy \\ -sy & +v \\ +t & \end{array}$$

Deinde pono ſingulos terminos $\infty 0$, adeò ut tum habeas has duas æquationes,

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- To get remainder 0, we need to make both coefficients 0 by finding a solution (y, w) to this system of equations:

$$-y^5 + 4y^3w - 3yw^2 + 1 = 0$$
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3. Thus $x^2 + yx + w \approx x^2 + 1.581334x + 0.715459$ is nearly a divisor. The remainder is

$$-1.318102 \times 10^{-7}x - 4.735616 \times 10^{-8}.$$

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- Making these notions completely clear had to wait for Lagrange and Gauss.

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Try degree 12: the "quadratic factor" system should reduce to solving a single equation of degree $\binom{12}{2} = 66$,which is not odd,but 66 is oddly even, which case we resolved! This step is the start of an induction argument due to Foncenex, a student of Lagrange (1759).

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- After Foncenex, there were further clarifications to the argument in the 18th century—see, especially, Note IX of Lagrange's *Traité de la résolution des équations numérique de tous les degrés* (1808), which seems to be where Galois learned the theory of equations.

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- Emil Artin also provided a well-known variant using Sylow's theorem instead of Foncenex's induction. This variant is popular in graduate-level algebra textbooks.

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Thanks for coming!