Cracking the Cubic:

Cardano, Controversy, and Creasing

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These images are from the Wikipedia articles on Niccolò Fontana Tartaglia and Gerolamo Cardano. Both images belong to the public domain.

Quadratic Equation

A brief history...

- 400 BC Babylonians
- 300 BC Euclid

323 - 283 BC

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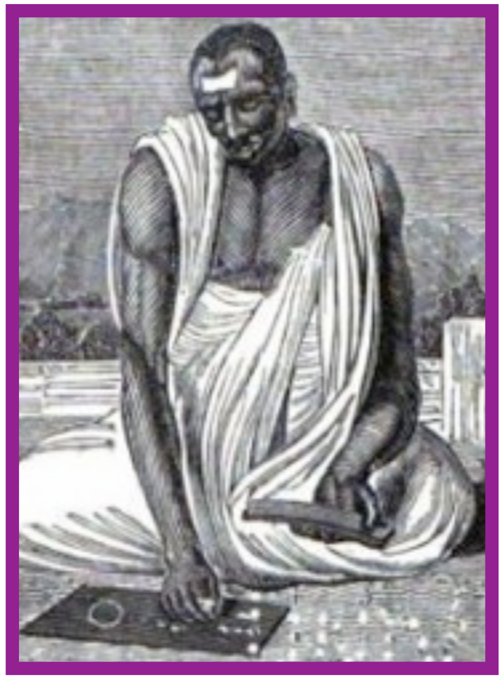
Quadratic Equation

- 600 AD Brahmagupta



598 - 668 BC

To the absolute number multiplied by four times the [coefficient of the] square, add the square of the [coefficient of the] middle term; the square root of the same, less the [coefficient of the] middle term, being divided by twice the [coefficient of the] square is the value.

Brahmasphutasiddhanta
Colebook translation, 1817, pg 346

$$ax^2 + bx = c$$

$$x = \frac{\sqrt{4ac + b^2} - b}{2a}$$

This image is from the website entry for Brahmagupta from the The Story of Mathematics. It belongs to the public domain.

Quadratic Equation

- 800 AD al-Khwarizmi
- 12th cent bar Hiyya (Savasorda)
  Liber embadorum
- 13th cent Yang Hui

780 - 850

This image is from the website entry for al-Khwarizmi from The Story of Mathematics. It belongs to the public domain.

Luca Pacioli

1445 - 1509

Summa de arithmetica, geometrica, proportioni et proportionalita (1494)

Cubic Equation

***Challenge**:* Solve the equation

$$ax^3 + bx^2 + cx + d = 0$$

The quest for the solution to the cubic begins!

Enter Scipione del Ferro...

Scipione del Ferro

- 1465 - 1526, Italian
- Chair of math dept at University of Bologna
- First to solve *depressed cubic:* $x^3 + mx = n$
- Kept formula secret!
- Revealed method to student *Antonio Fior* on deathbed

Nicolo of Brescia (Tartaglia)

- 1500 - 1557, Italian
- Feb 13, 1535 solved $x^3 + mx^2 = n$
- Won challenge!

This image is from the Wikipedia article on [Niccolò Fontana Tartaglia](). It belongs to the public domain.

Girolamo Cardano

- 1501 - 1576, Italian
- Numerous ailments when young
- Became a physician
- Wrote treatise on probability
- Brought Tartaglia to Milan to learn secret of the cubic

This image is from the Wikipedia article on [Gerolamo Cardano](). It belongs to the public domain.

The (encoded) solution!

When the cube and things together
Are equal to some discreet number,
Find two other numbers differing in this one.
Then you will keep this as a habit
That their product should always be equal
Exactly to the cube of a third of the things.
The remainder then as a general rule
Of their cube roots subtracted
Will be equal to your principal thing

The (encoded) solution!

In the second of these acts,
When the cube remains alone,
You will observe these other agreements:
You will at once divide the number into two parts
So that the one times the other produces clearly
The cube of the third of the things exactly.
Then of these two parts, as a habitual rule,
You will take the cube roots added together,
And this sum will be your thought.

The (encoded) solution!

The third of these calculations of ours
Is solved with the second if you take good care,
As in their nature they are almost matched.
These things I found, and not with sluggish steps,
In the year one thousand five hundred, four and thirty.
With foundations strong and sturdy
In the city girdled by the sea.

The (encoded) solution!

This verse speaks so clearly that, without any other example, I believe that your Excellency will understand everything. - Tartaglia

I swear to you, by God's holy Gospels, and as a true man of honour, not only never to publish your discoveries, if you teach me them, but I also promise you, and I pledge my faith as a true Christian, to note them down in code, so that after my death no one will be able to understand them. - Cardano

Lodovico Ferrari

- 1522 - 1565, Italian
- Started out as Cardano's servant
- Quickly became colleagues
- Cardano reveals Tartaglia's secret solution
- Together solved general cubic and quartic!

Cardano and Ferrari

- Due to oath, could not publish their work!
- Traveled to Bologna seeking del Ferro's original work (1543)
- Found solution to depressed cubic!
- Cardano publishes *Ars Magna* in 1545
- Chapter XI "*On the Cube and First Power Equal to the Number*"

Ars Magna

In our own days Scipione del Ferro of Bologna has solved the case of the cube and first power equal to a constant, a very elegant and admirable accomplishment...In emulation of him, my friend Niccolo Tartaglia of Brescia, wanting not to be outdone, solved the same case when he got into a contest with his [Scipione's] pupil, Antonio Maria Fior, and, moved by my many entreaties, gave it to me.

Ars Magna

For I had been deceived by the world of Luca Paccioli, who denied that any more general rule could be discovered than his own. Notwithstanding the many things which I had already discovered, as is well known, I had despaired and had not attempted to look any further. Then, however, having received Tartaglia's solution and seeking for the proof of it, I came to understand that there were a great many other things that could also be had. Pursuing this thought and with increased confidence, I discovered these others, partly by myself and partly through Lodovico Ferrari, formerly my pupil.

Ferrari vs. Tartaglia

- Public debate on August 10, 1548
- Refereed by Governor of Milan
- Each posed 62 problems
- Ferrari wins

There is a right-angled triangle, such that when the perpendicular is drawn, one of the sides with the opposite part of the base makes 30, and the other side with the other part makes 28. What is the length of one of the sides?

Cardano's Solution

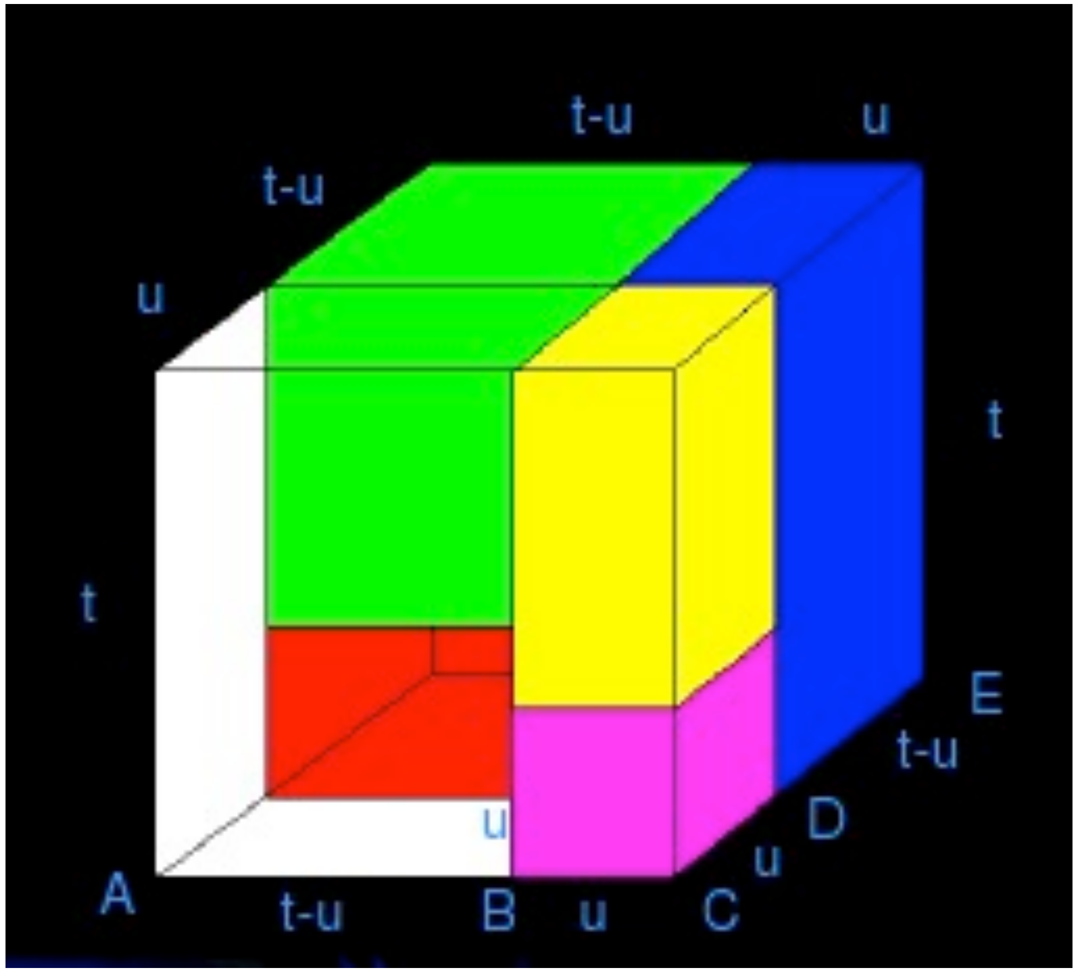
Method to solve $x^3 + mx = n$:

Cube one-third the coefficient of x; add to it the square of one-half the constant of the equation; and take the square root of the whole. You will duplicate [repeat] this, and to one of the two you add one-half the number you have already squared and from the other you subtract one-half the same. Then, subtracting the cube root of the first from the cube root of the second, the remainder which is left is the value of x.

Cardano's Solution

$$x = \sqrt[3]{\frac{n}{2} + \sqrt{\frac{n^2}{4} + \frac{m^3}{27}}} - \sqrt[3]{\frac{-n}{2} + \sqrt{\frac{n^2}{4} + \frac{m^3}{27}}}$$

Cardano's Solution



Vol of Pink Cube $= u^3$

Vol of Green Cube $= (t - u)^3$

Vol of Clear and Blue Slabs

$= 2tu(t - u)$

Vol of Yellow Block $= u^2(t - u)$

Vol of Red Block $= u(t - u)^2$

Total Volume (simplified):

$$t^3 - u^3 = (t - u)^3 + 3tu(t - u)$$

Cardano's Solution

Make a clever substitution in:

$$t^3 - u^3 = (t - u)^3 + 3tu(t - u)$$

Let $x = t - u$ to obtain:

$$x^3 + 3tux = t^3 - u^3$$

This is depressed where $m = 3tu$ and $n = t^3 - u^3$.

Solving for u in the first gives $u = m/3t$ and substituting this into the second gives:

$$n = t^3 - m^3/27t^3$$

Cardano's Solution

Multiplying $n = t^3 - m^3/27t^3$ by t^3 produces:

$$t^6 - nt^3 - m^3/27 = 0$$

which we can rewrite as:

$$(t^3)^2 - n(t^3) - m^3/27 = 0$$

This is a quadratic!!

Cardano's Solution

$$(t^3)^2 - n(t^3) - m^3/27 = 0$$

The quadratic formula gives solutions for t. Then we use $n = t^3 - u^3$ to solve for u and finally use $x = t - u$ to solve for x. Thus, we have:

$$x = \sqrt[3]{\frac{n}{2} + \sqrt{\frac{n^2}{4} + \frac{m^3}{27}}} - \sqrt[3]{\frac{-n}{2} + \sqrt{\frac{n^2}{4} + \frac{m^3}{27}}}$$

Ars Magna

Chapter XI: example illustrating technique for $x^3 + 6x = 20$

Chapter XII: solved $x^3 = mx + n$

Chapter XIII: solved $x^3 + n = mx$

But what about the *general* cubic:

$$ax^3 + bx^2 + cx + d = 0$$

General Cubic

$$ax^3 + bx^2 + cx + d = 0$$

The key is to make a clever substitution:

$$x = y - b/3a$$

This results in a depressed equation

$$y^3 + py = q$$

Negative Roots

Puzzle: But what about negative roots?

Example: Find the roots of $x^3 - 15x = 4$

$$x = \sqrt[3]{2+\sqrt{-121}} - \sqrt[3]{-2+\sqrt{-121}}$$

Negative Roots

Rafael Bombelli
1526 - 1573

$$
\begin{aligned}
(2+\sqrt{-1})^3 &= 8+12\sqrt{-1}-6-\sqrt{-1}\\
&= 2+11\sqrt{-1}\\
&= 2+\sqrt{-121}
\end{aligned}
$$

$$x = \sqrt[3]{2+\sqrt{-121}} - \sqrt[3]{-2+\sqrt{-121}}$$

This image is from the website entry for Rafael Bombelli from the [MacTutor History of Mathematics](). It belongs to the public domain.

Negative Roots

Rafael Bombelli
1526 - 1573

Plus by plus of minus, makes plus of minus.
Minus by plus of minus, makes minus of minus.
Plus by minus of minus, makes minus of minus.
Minus by minus of minus, makes plus of minus.
Plus of minus by plus of minus, makes minus.
Plus of minus by minus of minus, makes plus.
Minus of minus by plus of minus, makes plus.
Minus of minus by minus of minus makes minus.

$\sqrt{-x}$ = *"plus of minus"*

Quartic Equation

Puzzle: What about the quartic?

$$ax^4 + bx^3 + cx^2 + dx + e = 0$$

Step One: Divide by a and make a substitution to obtain a depressed equation:

$$y^4 + my^2 + ny = p$$

Step Two: Replace this by a related cubic, then use previous techniques.

Origami Solution

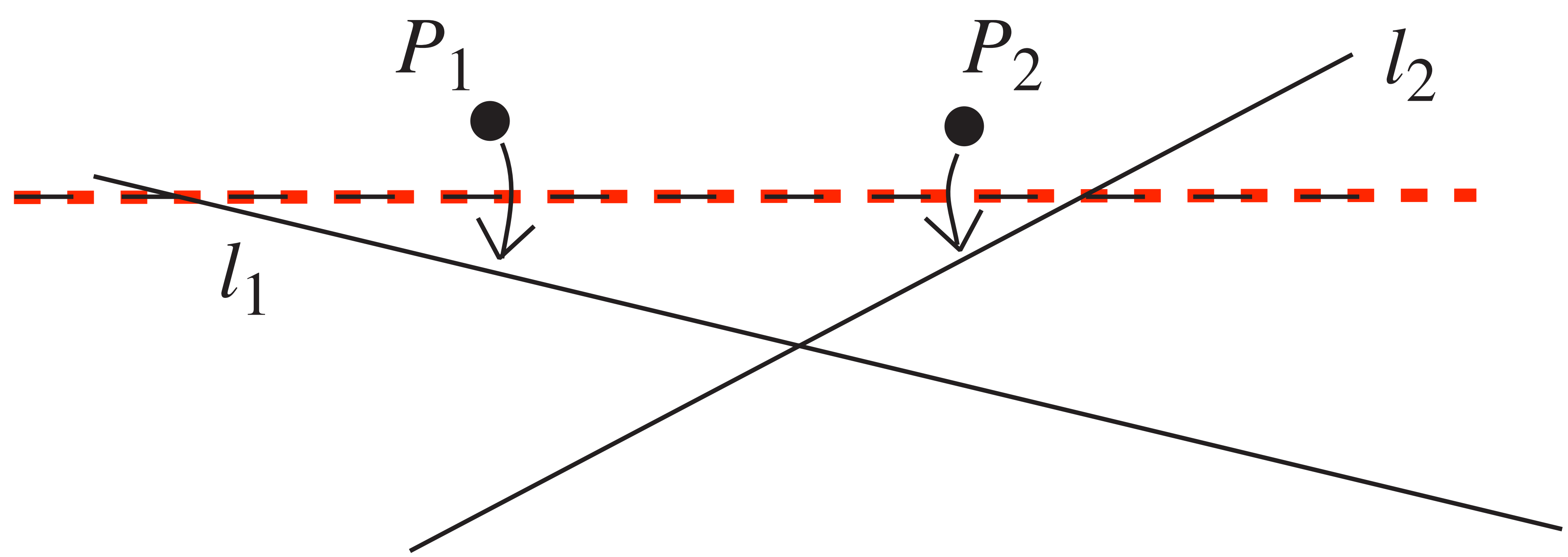
Elementary Moves:

Given two points P and Q, we can make a crease line that places P onto Q when folded.

Given a line l and point P not on l, we can make a crease line that passes through P and is perpendicular to l.

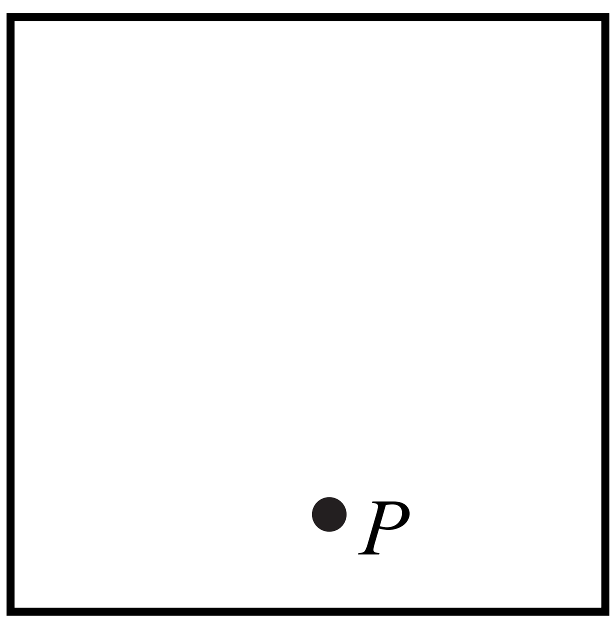
Beloch Fold

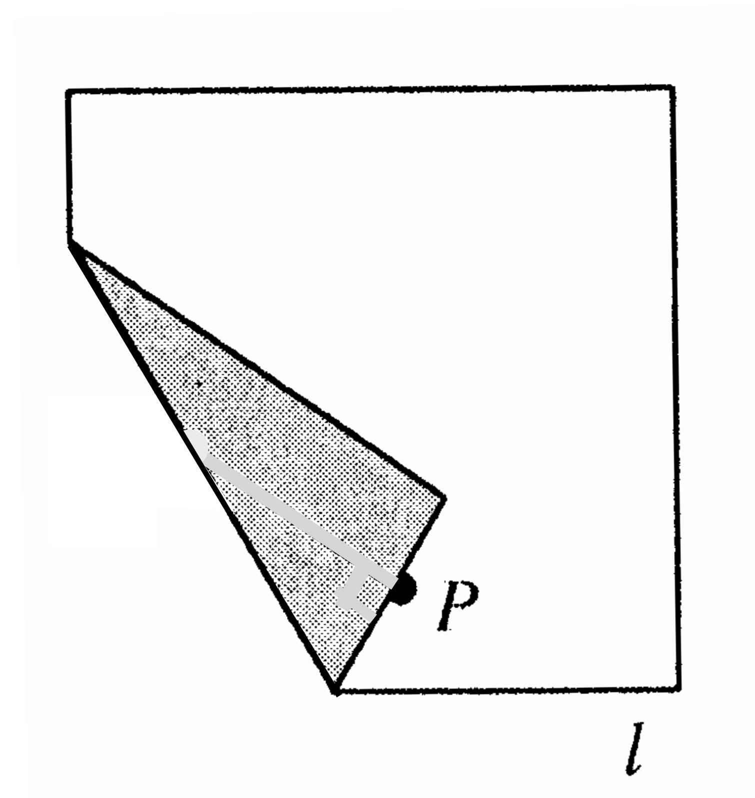
Given two points P_1 and P_2 and two lines l_1 and l_2 we can, whenever possible, make a single fold that places P_1 onto l_1 and P_2 onto l_2 simultaneously.



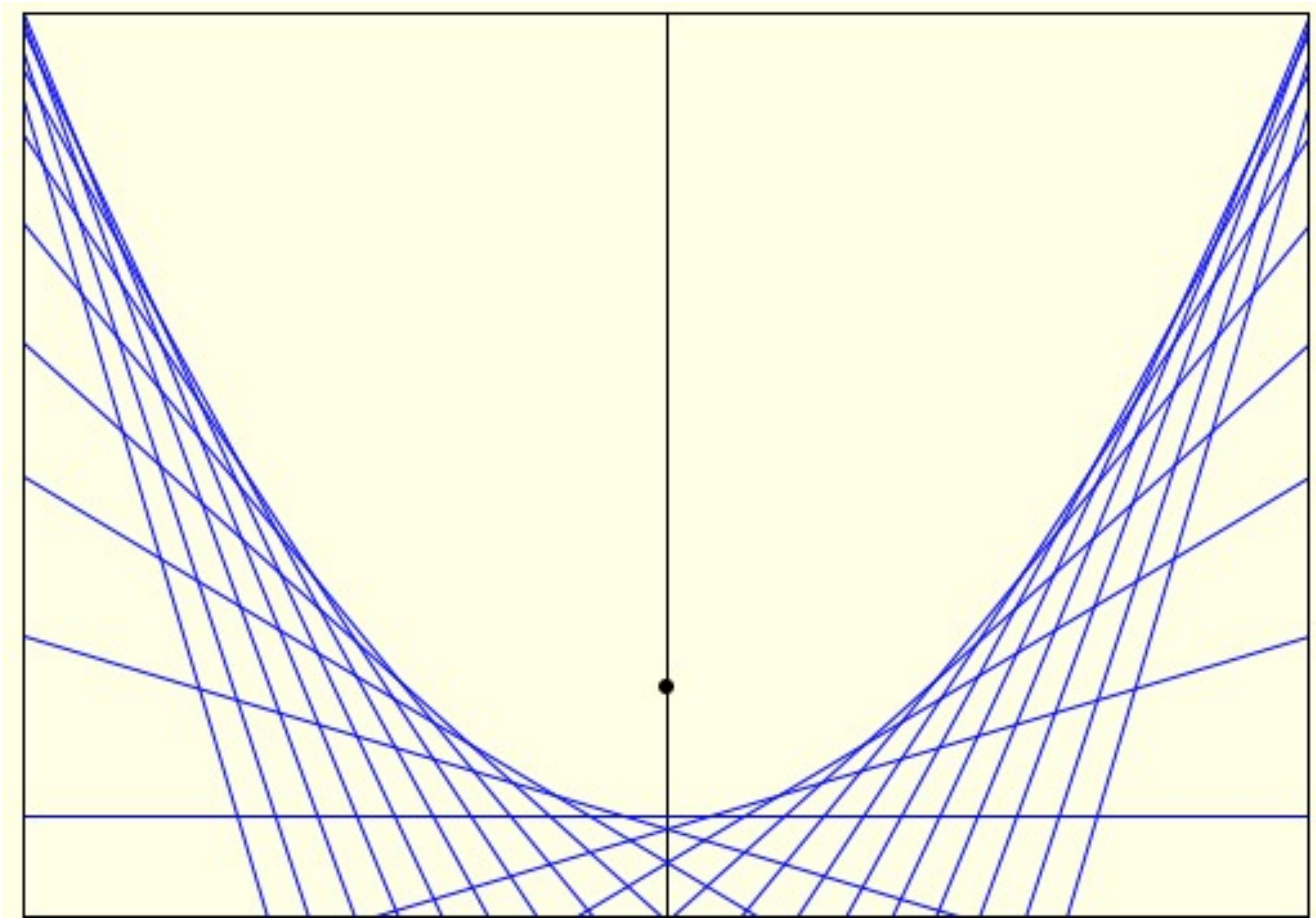
Beloch Fold

What is this fold accomplishing?





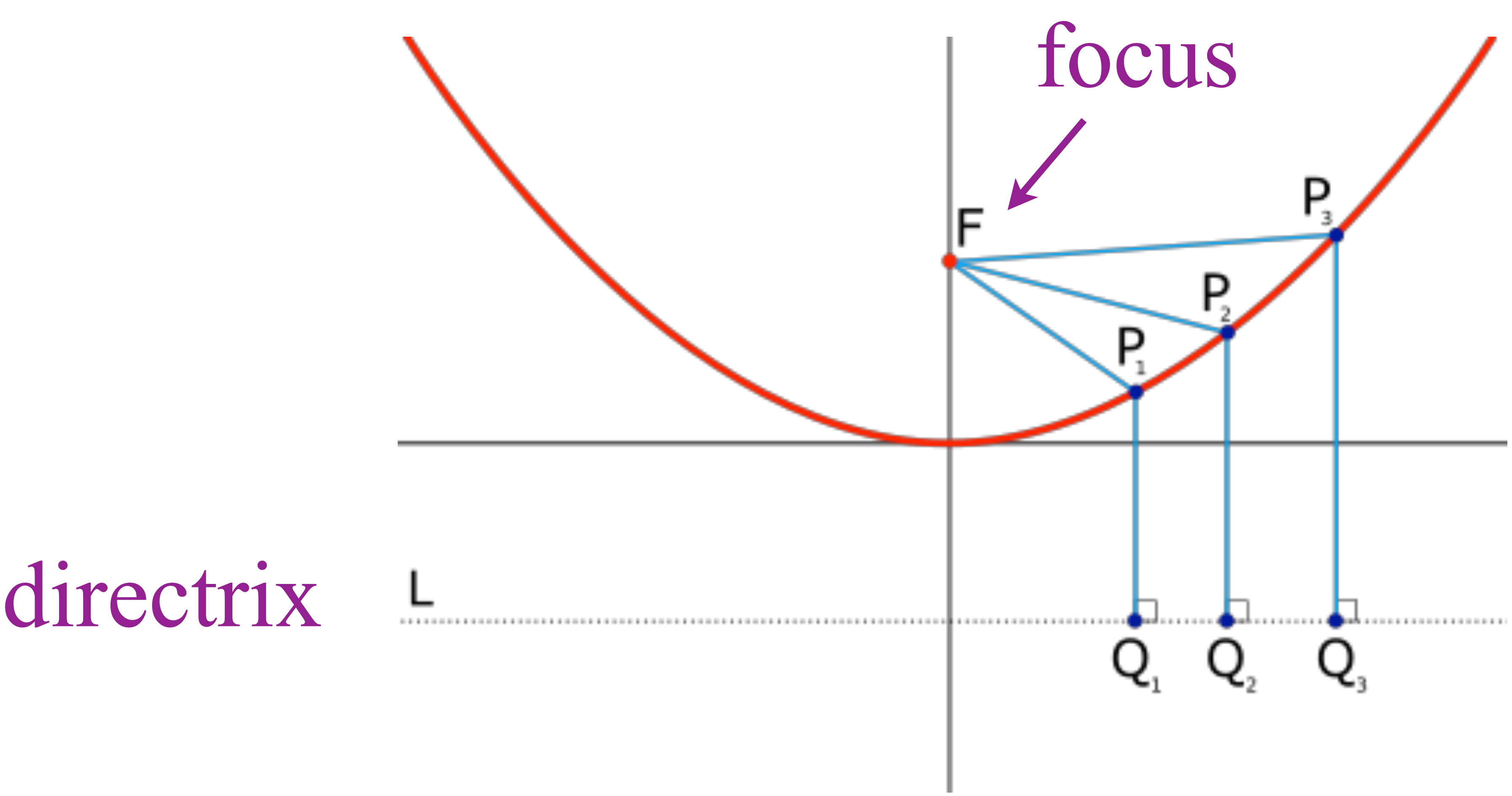
Beloch Fold



These crease lines are tangent to the parabola with focus P and directrix l.

This image was created using the java applet from the website Cut The Knot.

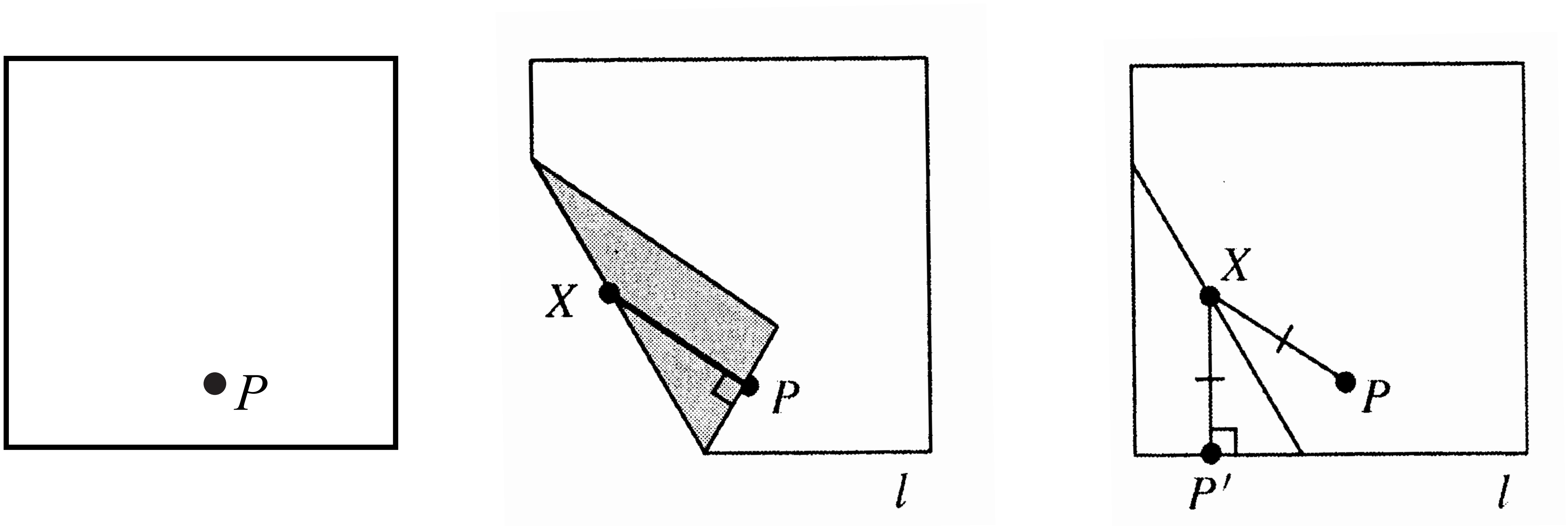
Review of Parabolas



This image is from the Wikipedia article on parabola. It belongs to the public domain.

Beloch Fold

Picture Proof:



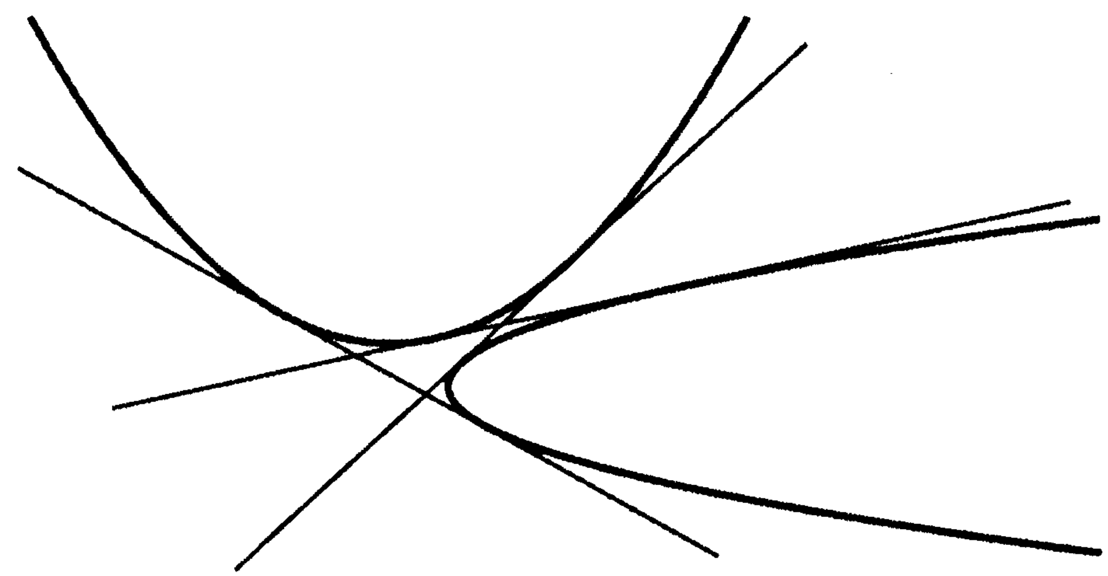
The Beloch fold finds a common tangent to two parabolas!

Beloch Fold

Morals:

Folding a point to a line is equivalent to solving a quadratic equation.

The Beloch fold, then, is equivalent to solving a cubic equation.



Margherita Piazzolla Beloch

- 1879 - 1976, Italian
- Algebraic geometer, Chair at Univ. of Ferrara
- First to discover origami can find common tangents to two parabolas!
- Beloch fold is most complicated paper-folding move possible

Margherita Piazzolla Beloch

Sul metodo del ripiegamento della carta per la risoluzione dei problemi geometrici

Margherita Piazzolla Beloch
(original Italian version)

1. Stralcio dalle mie lezioni del corso di Matematiche complementari, tenuto all' Universita di Ferrara nell' anno accademico 1933-34, alcune osservazioni sul *Metodo del ripiegamento della carta*[1] osservazioni che valgono a dare maggiore portata a questo metodo, sia come mezzo di risoluzione effettiva di alcuni problemi, sia come semplicità di costruzione in confronto delle costruzioni con riga e compasso dal punto di vista della geometrografia.

2. Il primo ad attirare, col suo autorevole giudizio, l'attenzione degli studiosi sul metodo del ripiegamento della carta dovuto al matematico Indiano Sundara Row, fu il Klein nelle sue celebri Conferenze su questioni di matematica.[2]

Ora si può osservare che questo metodo più che una semplice curiosità matematica, costituisca uno strumento che può servire utilmente per la risoluzione effettiva di una vasta categoria di problemi geometrici non risolubili con riga e compasso, come pure può rappresentare un effettivo risparmio di tempo per certe costruzioni risolubili con riga e compasso come per es, in determinate costruzioni per tangenti delle coniche[3] che possono utilmente servire per procurarsi con poca fatica dei modelli delle tre specie coniche.

Fin dall'antichità furono posti accanto a riga e compasso altri strumenti per quei problemi la cui risoluzione era stata tentata invano con riga e compasso, Questi strumenti però non si trovano alla portata di tutti, sebbene

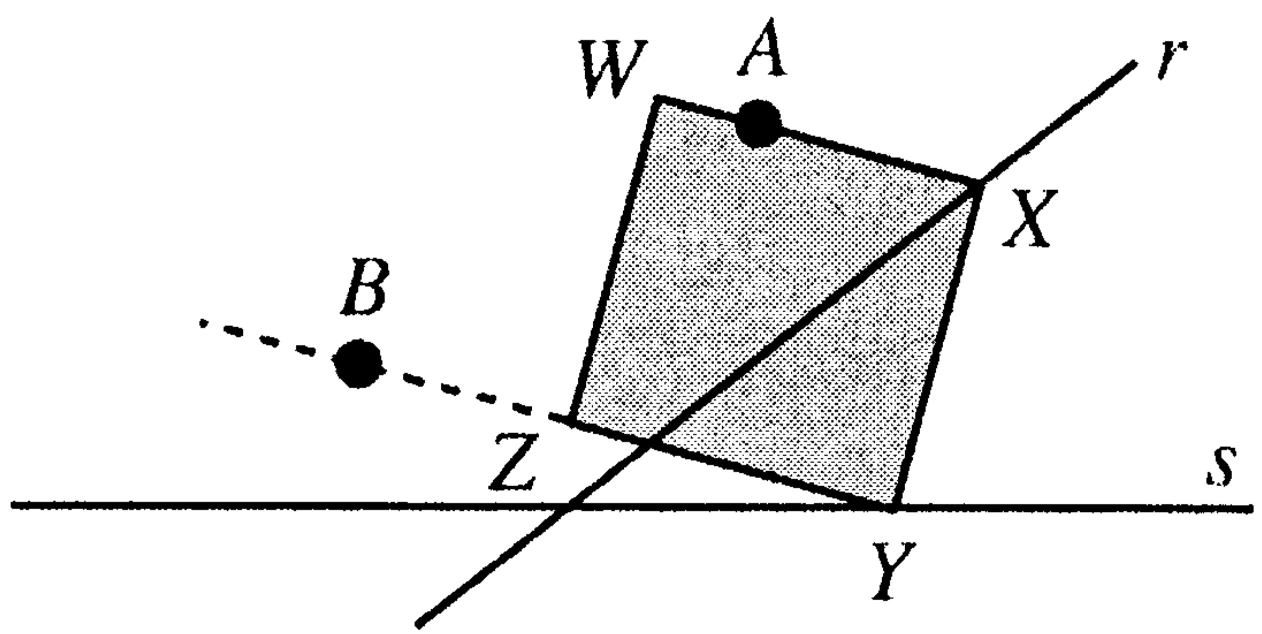
[1]Vedi Sundara-Row: *Geometric Exercises in Paper Folding* (Madras, Addison & C., 1893; Court Company, 1917).

[2]F. Klein: *Elementarmathematik von höheren Standpunkte aus*, Bd. II, p. 267, (Berlin, Springer, 1926).

[3]v. loc. cit., e C. A. Rupp: *On a transformation by paper-folding* (Amer. Math. Monthly, vol. XXXI, 1924, p. 432).

Beloch Square

Beloch Square: Given two points A and B and two lines r and s in the plane, construct a square $WXYZ$ with two adjacent corners X and Y lying on r and s, respectively, and the sides WX and YZ, or their extensions, passing through A and B, respectively.



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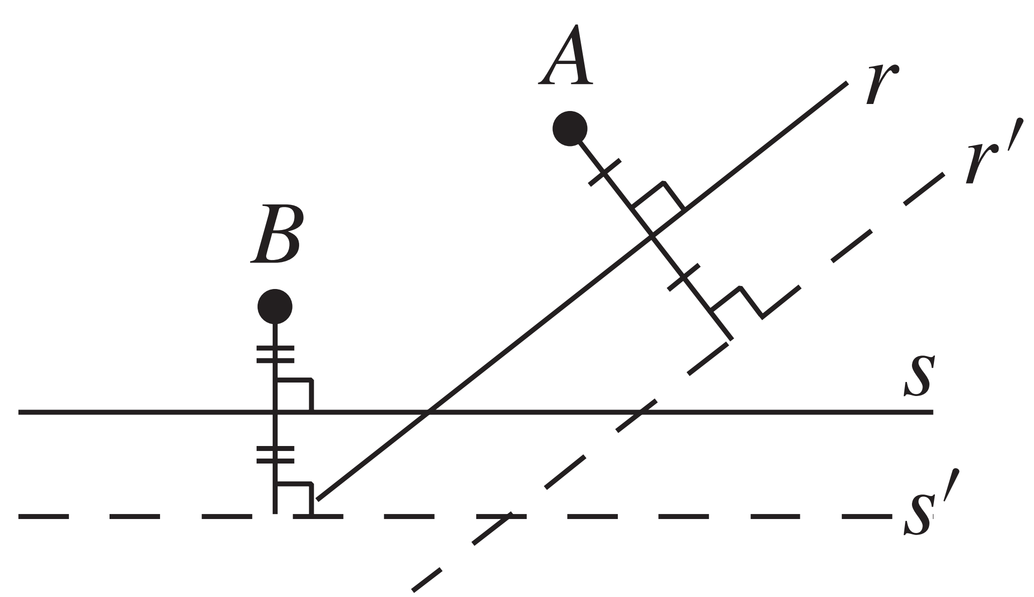

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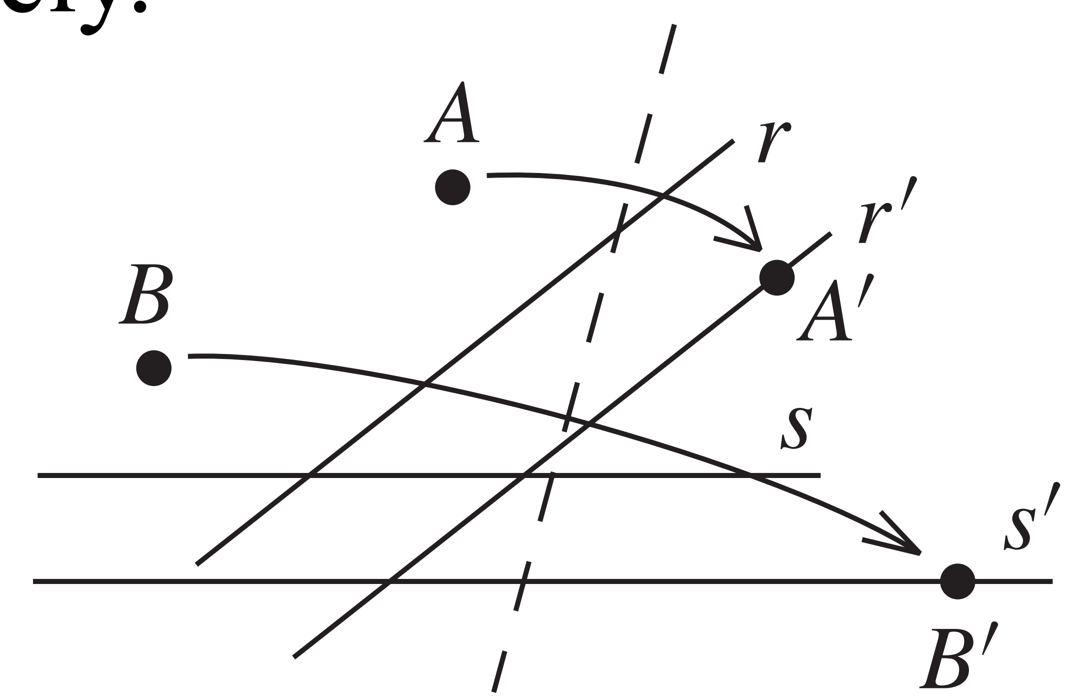

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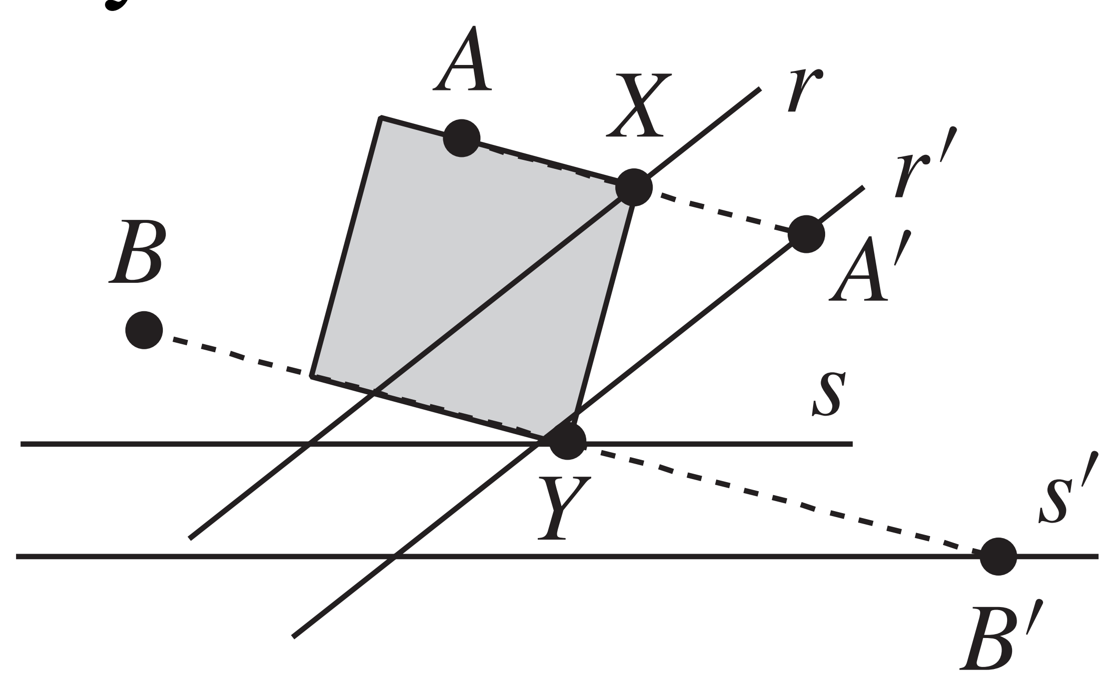
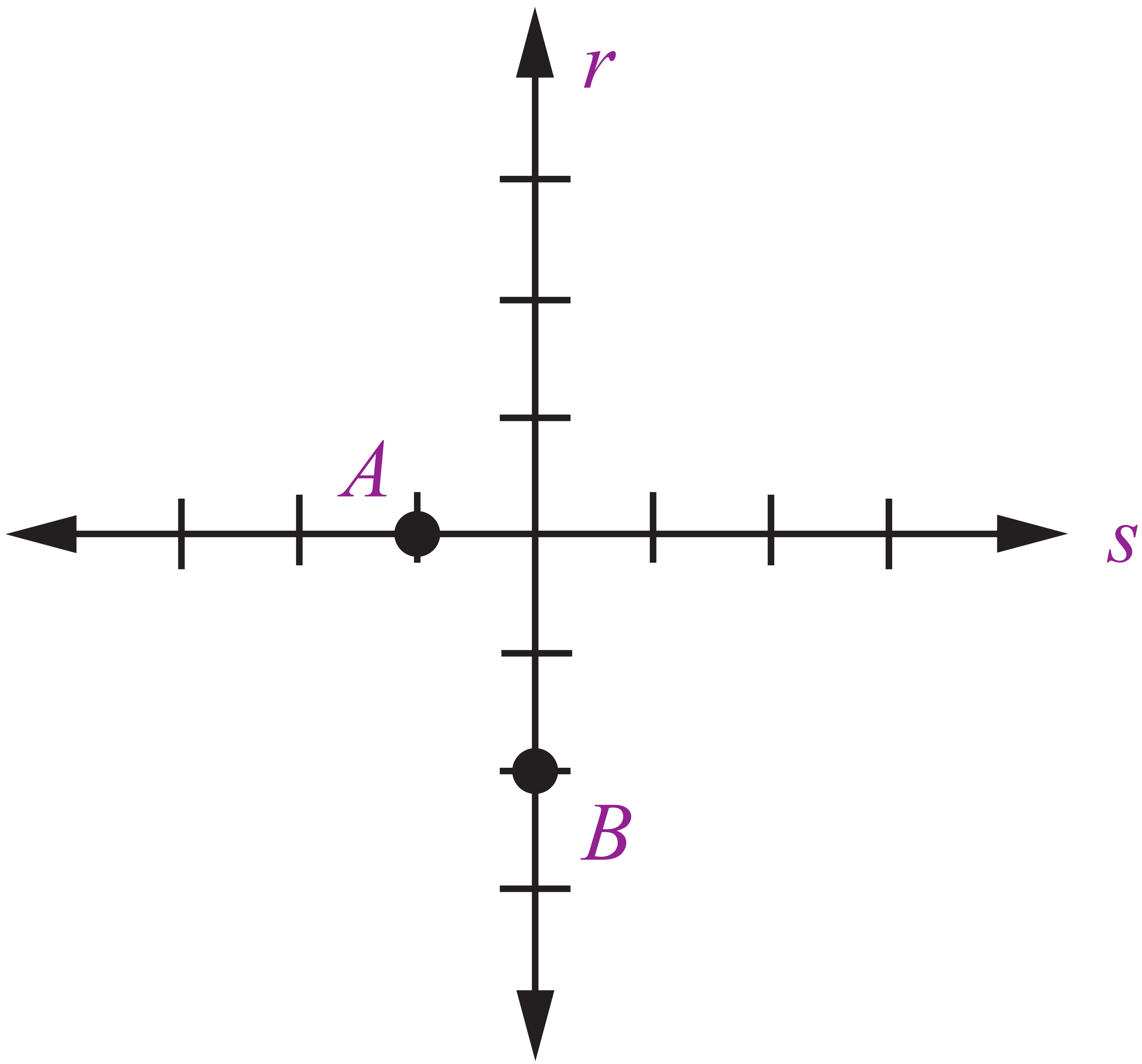


Constructing the Cube Root of 2



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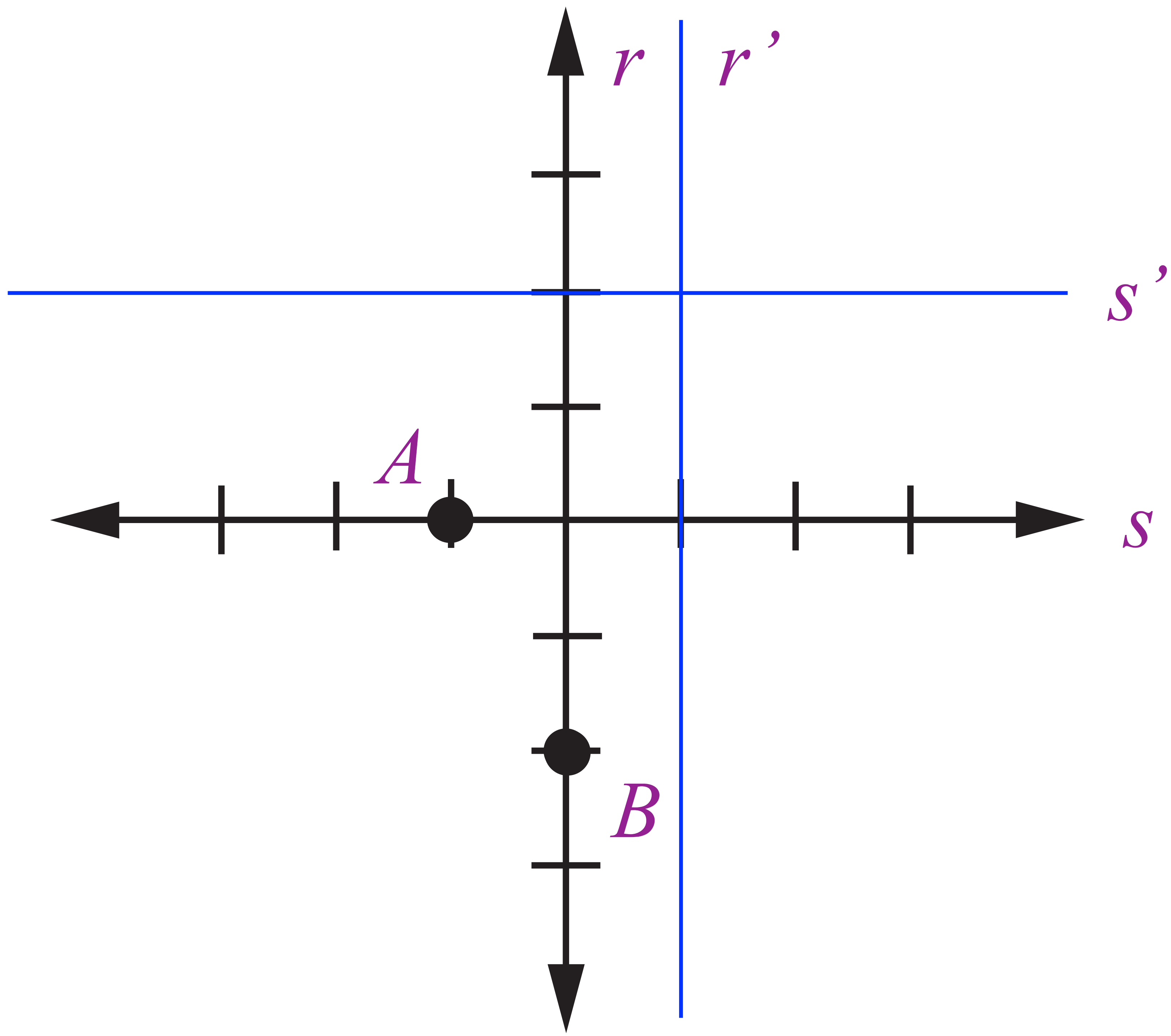

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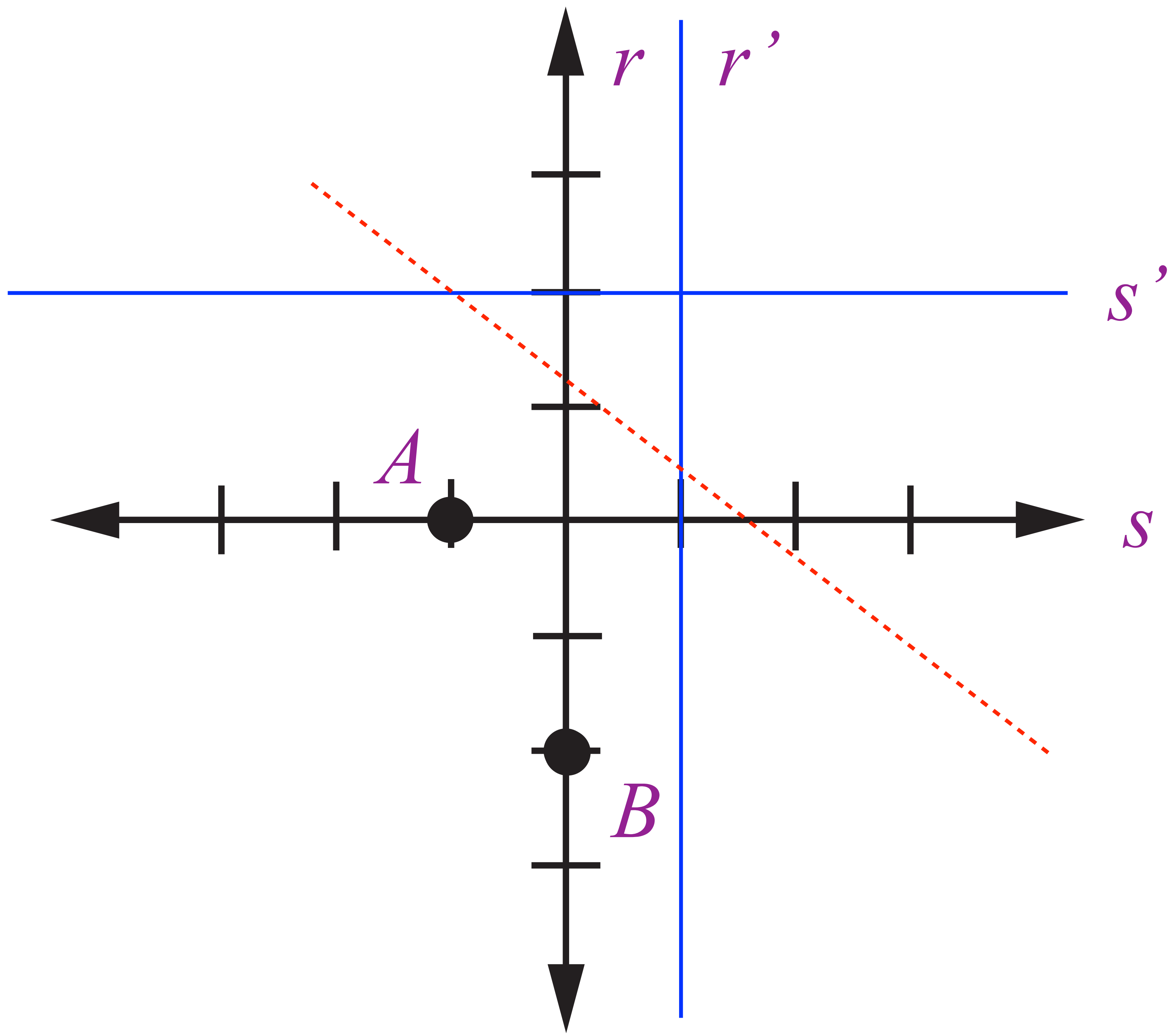

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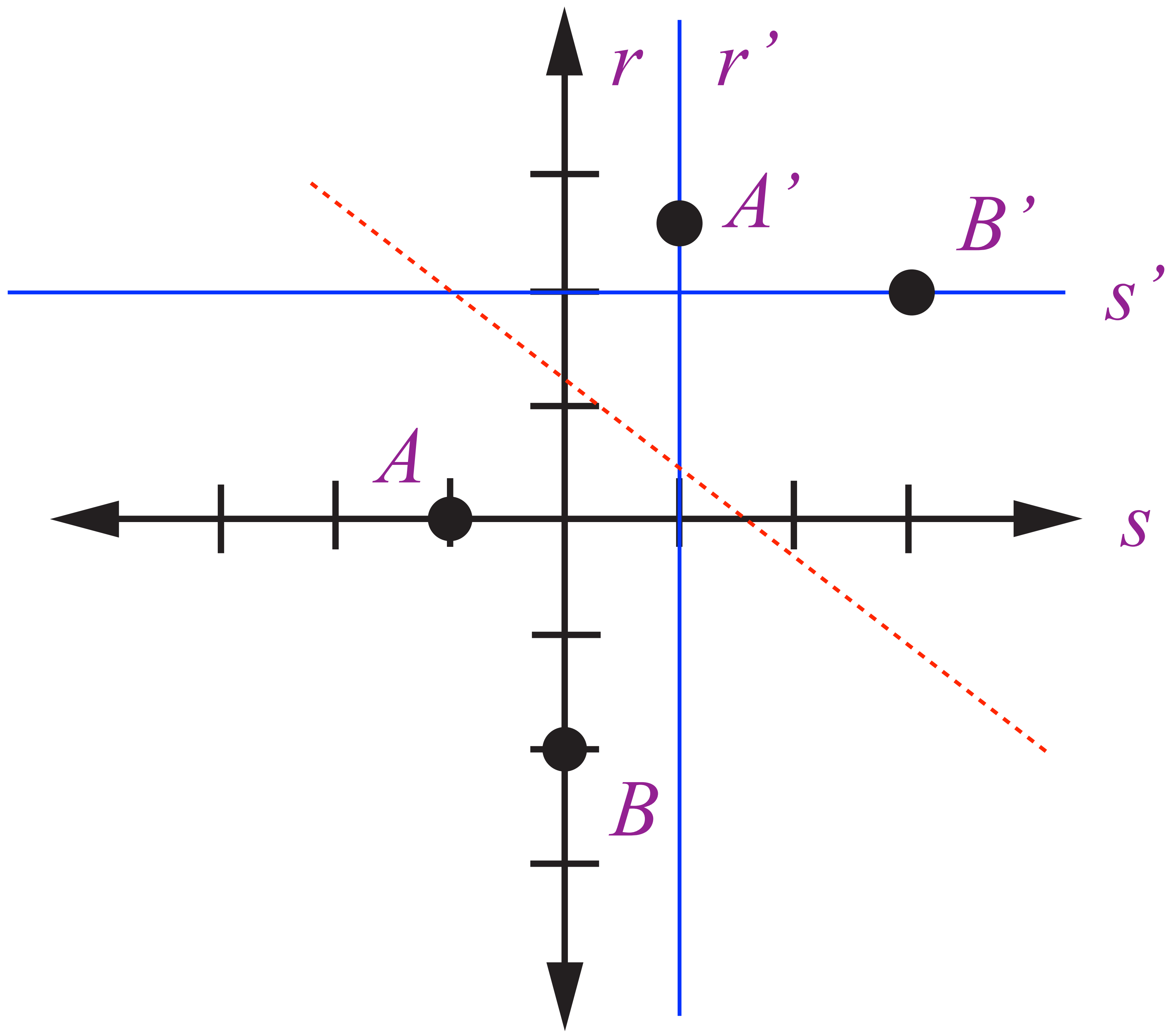

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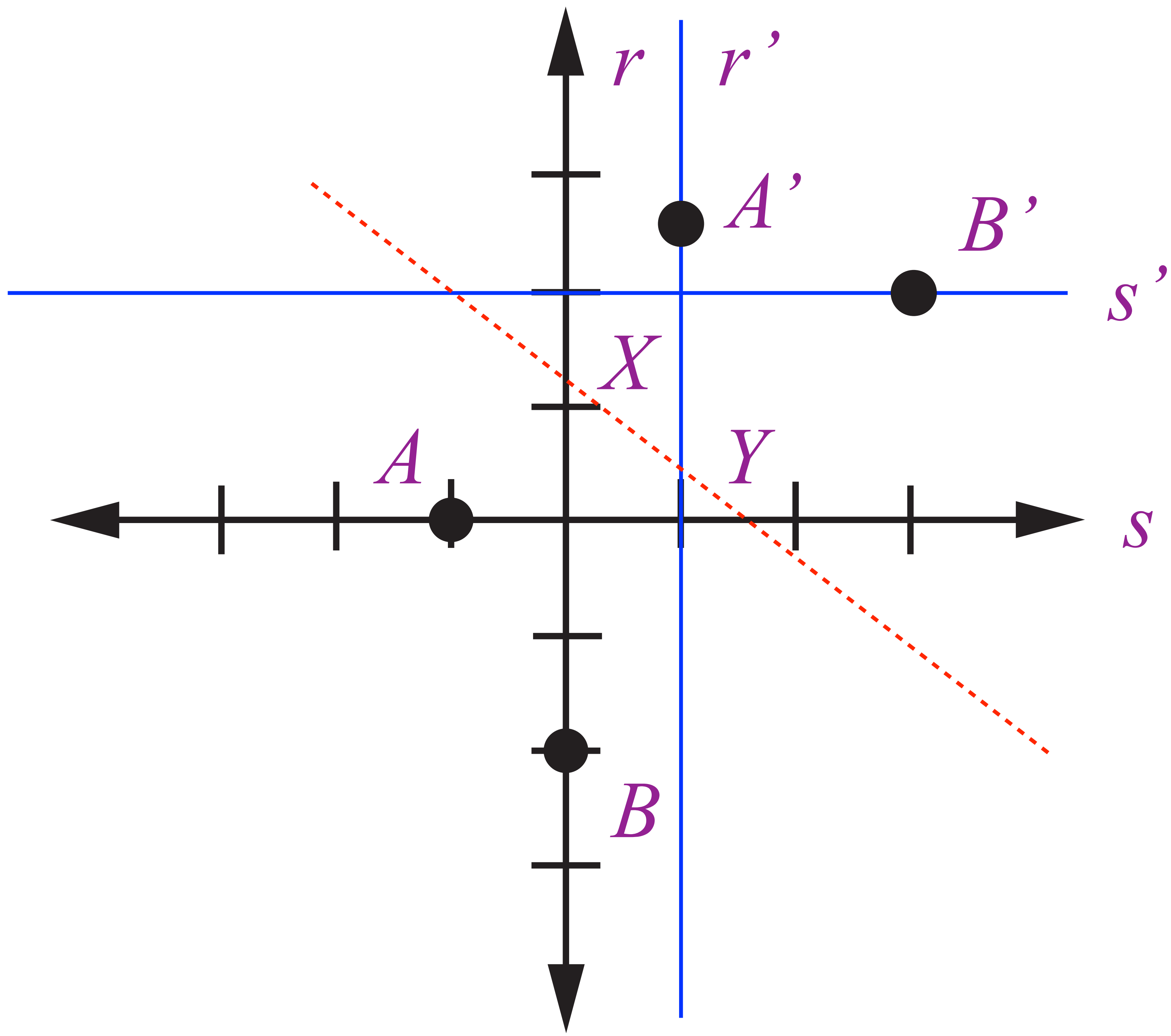

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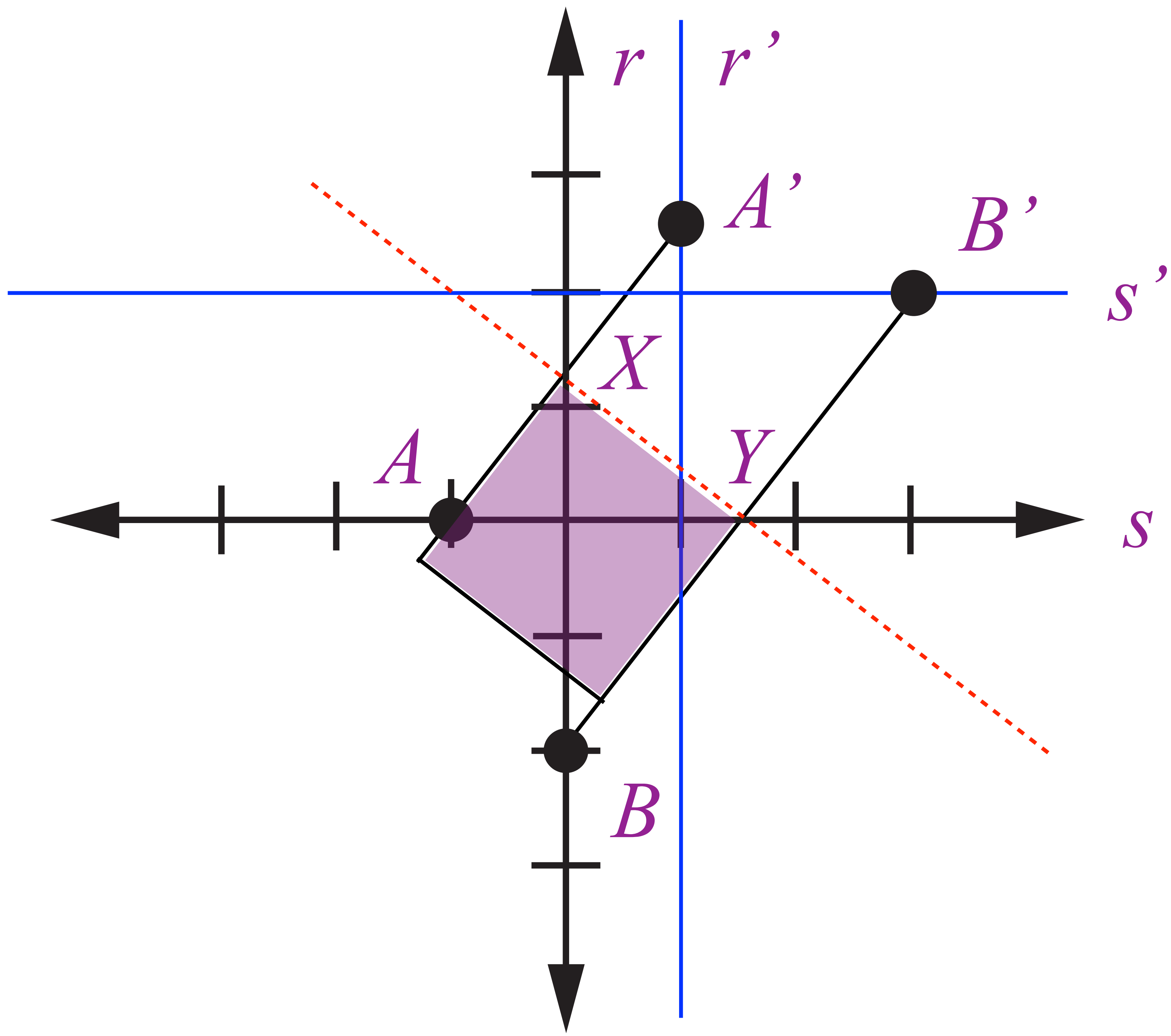

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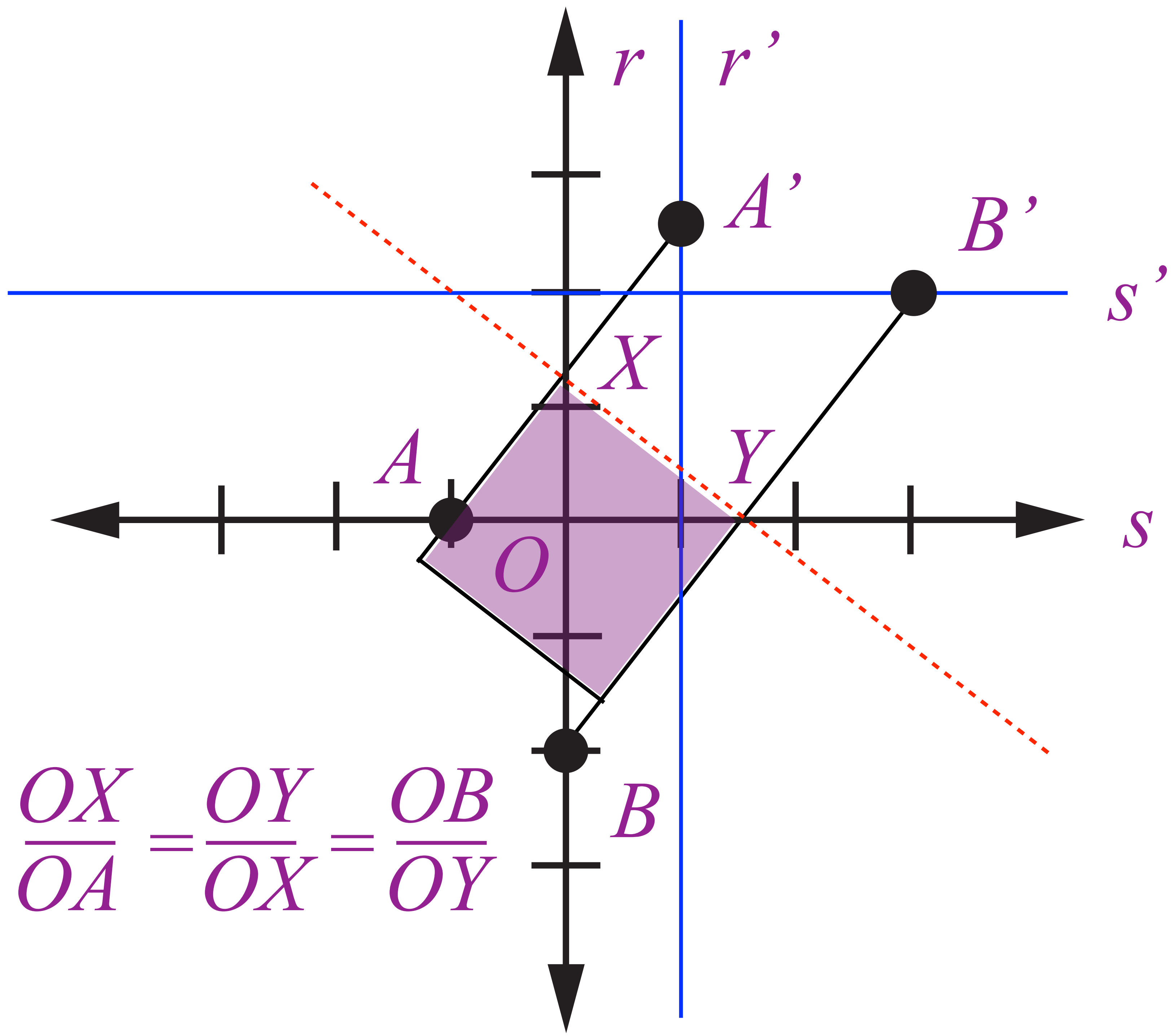


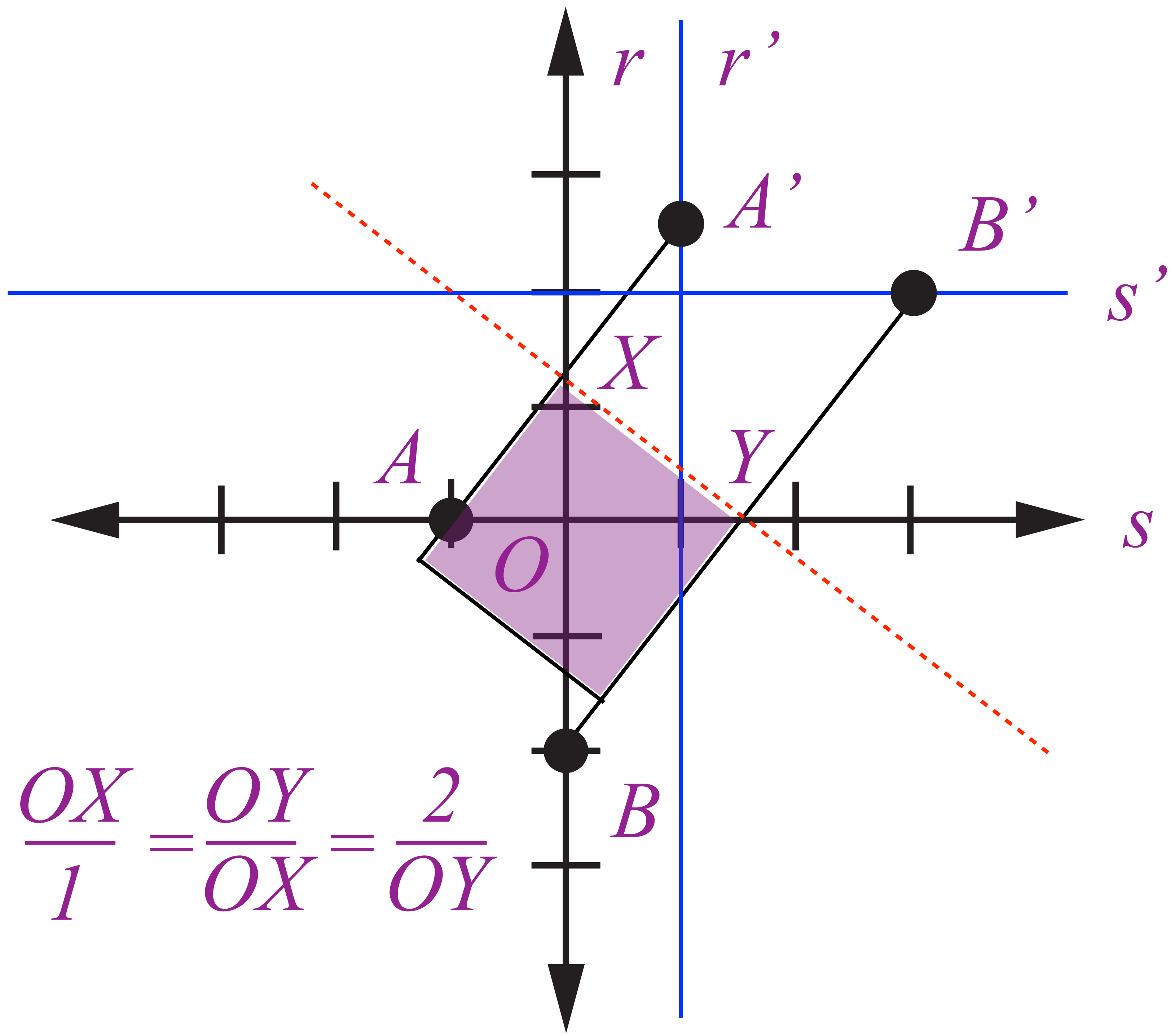
$\triangle OAX$

$\triangle OXY$

$\triangle OBY$

$$\frac{OX}{OA} = \frac{OY}{OX} = \frac{OB}{OY}$$

Constructing the Cube Root of 2



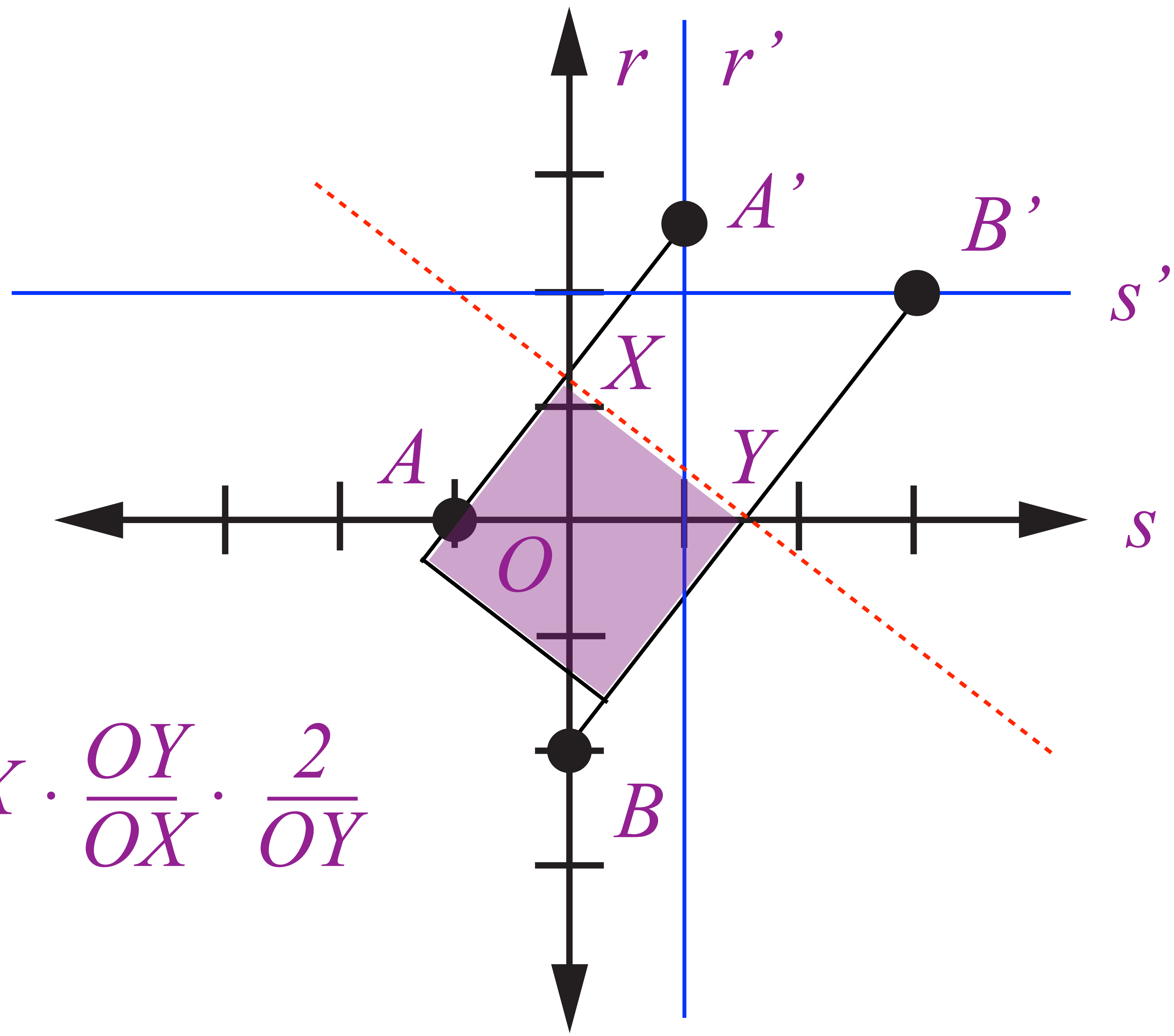
$\triangle OAX$

$\triangle OXY$

$\triangle OBY$

$$\frac{OX}{1} = \frac{OY}{OX} = \frac{2}{OY}$$

Constructing the Cube Root of 2

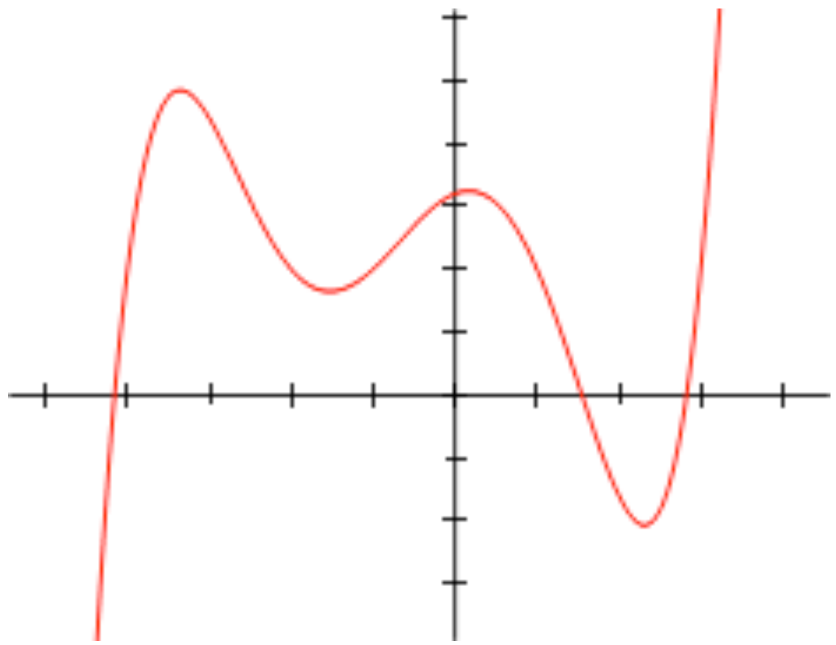


$$(OX)^3 = OX \cdot \frac{OY}{OX} \cdot \frac{2}{OY}$$

$$= 2$$

Epilogue

Puzzle: What about the quintic?



Does there exist a "*solution by radicals,*" that is, a formula for its roots that involves only the original coefficients and the algebraic operations of addition, subtraction, multiplication and division?

Epilogue

Paolo Ruffini
1765 - 1822

- 250 years since quartic solved
- 1790's sends work to Lagrange

The algebraic solution of general equations of degree greater than four is always impossible. Behold a very important theorem which I believe I am able to assert (if I do not err): to present the proof of it is the main reason for publishing this volume. The immortal Lagrange, with his sublime reflections, has provided the basis of my proof.

Epilogue

Paolo Ruffini
1765 - 1822

- 250 years since quartic solved
- 1790's sends work to Lagrange
- Sends work to Institute of Paris and Royal Society

... if a thing is not of importance, no notice is taken of it and Lagrange himself, "with his coolness" found little in it worthy of attention.

This image is from the Wikipedia article on [Paolo Ruffini](). It belongs to the public domain.

Epilogue

Geometers have occupied themselves a great deal with the general solution of algebraic equations and several among them have sought to prove the impossibility. But, if I am not mistaken, they have not succeeded up to the present. (1824)

Niels Abel
1802 - 1829

Why does Abel get the credit?

This image is from the Wikipedia article on Niels Henrik Abel. It belongs to the public domain.

Epilogue

... the mathematical community was not ready to accept so revolutionary an idea: that a polynomial could not be solved in radicals. Then, too, the method of permutations was too exotic and, it must be conceded, Ruffini's early account is not easy to follow. ... between 1800 and 1820 say, the mood of the mathematical community ... changed from one attempting to solve the quintic to one proving its impossibility...

References

- Dunham, W. *Journey through Genius: The Great Theorems of Mathematics,* John Wiley & Sons: New York, 1990, 133 - 154
- Hull, T. "Solving Cubics with Creases: The Work of Beloch and Lill," *American Mathematical Monthly* Vol. 118, No. 4 (April 2011), 307 - 315
- MacTutor History of Mathematics