# An extension of the Pythagorean theorem, with applications


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# Overview

# Outline

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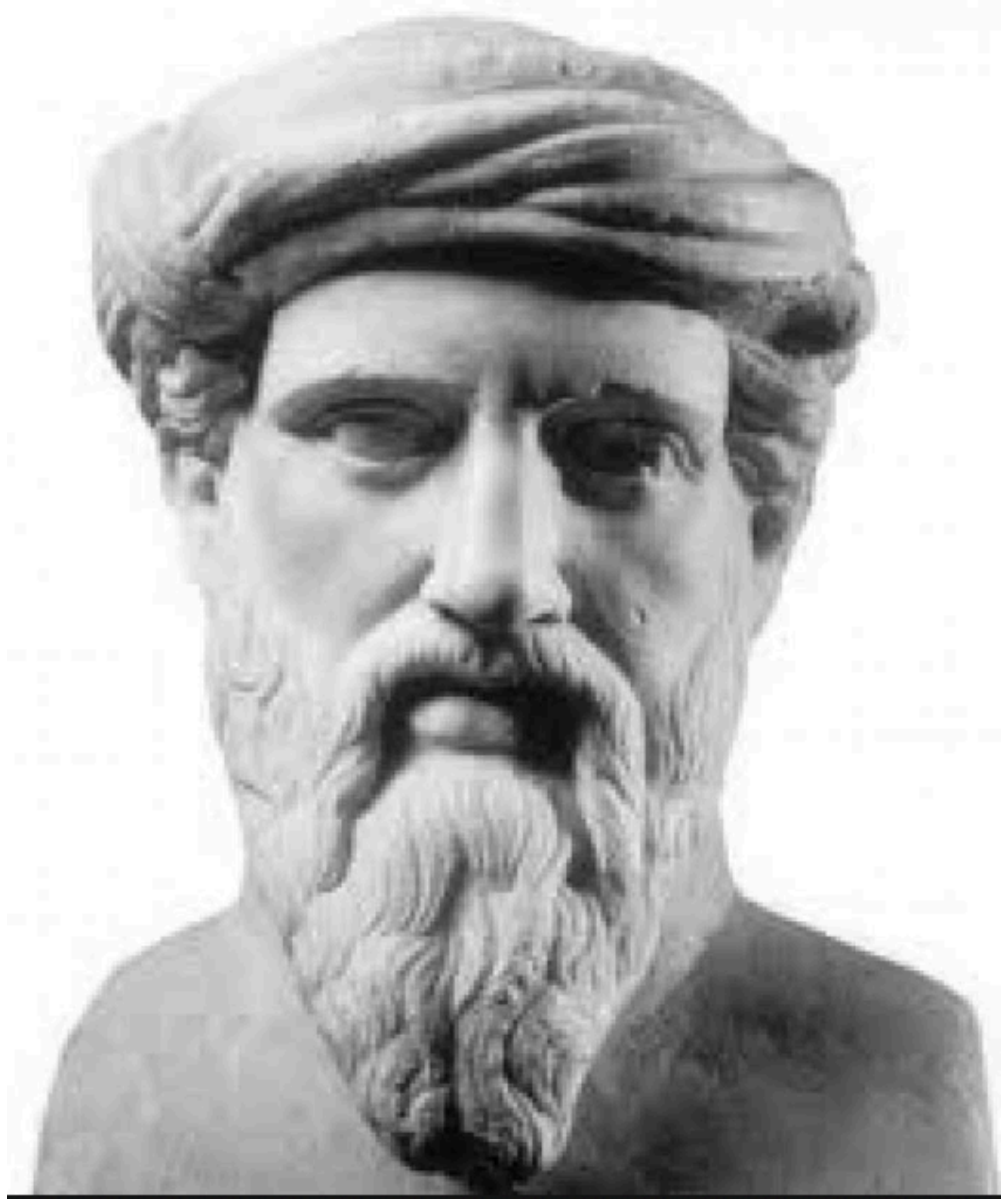
Review of the Pythagorean theorem

Orthogonality in normed spaces

Extension of the Pythagorean theorem

Applications



# Review of the Pythagorean theorem

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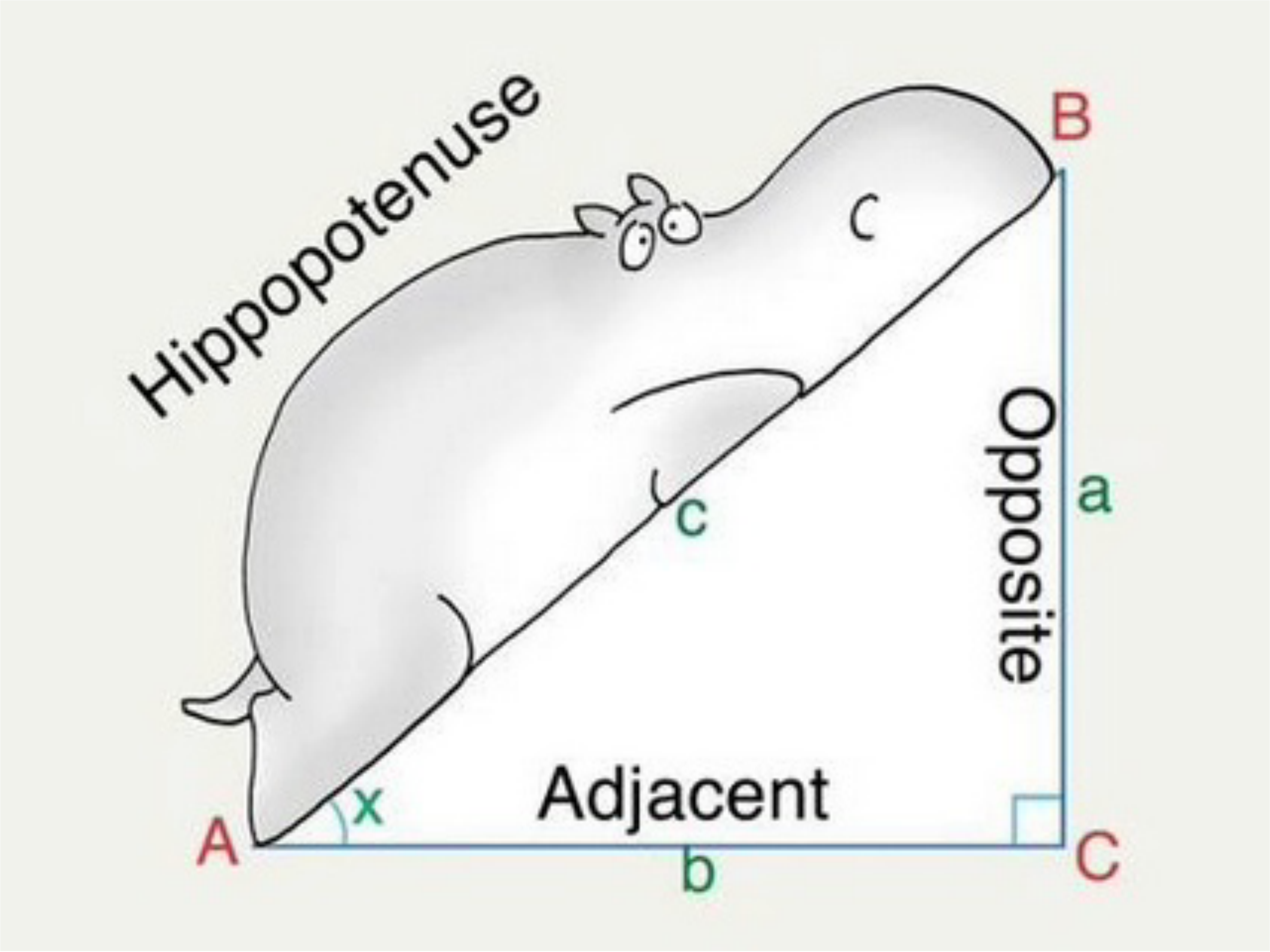
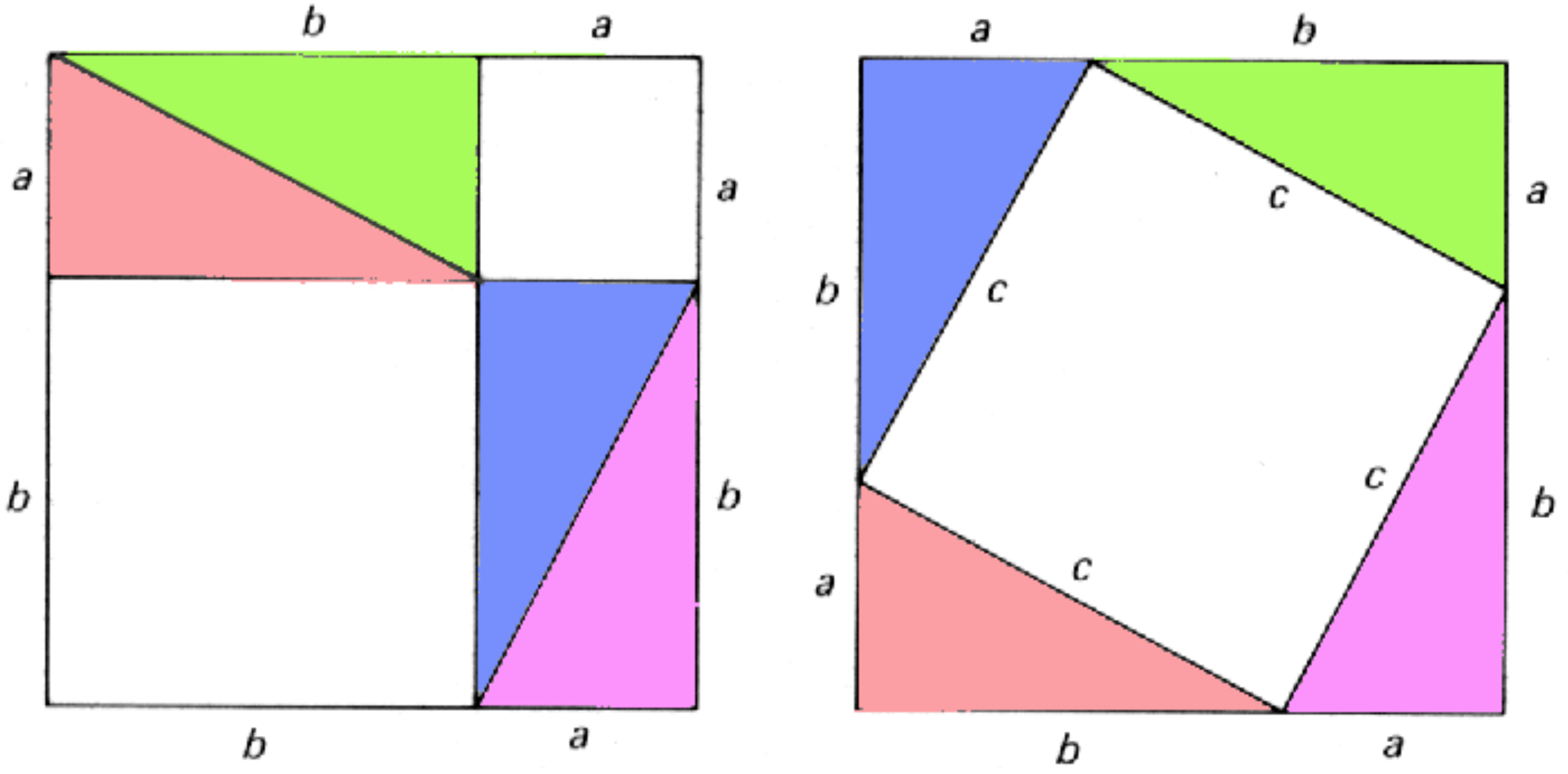


# Review of the Pythagorean theorem



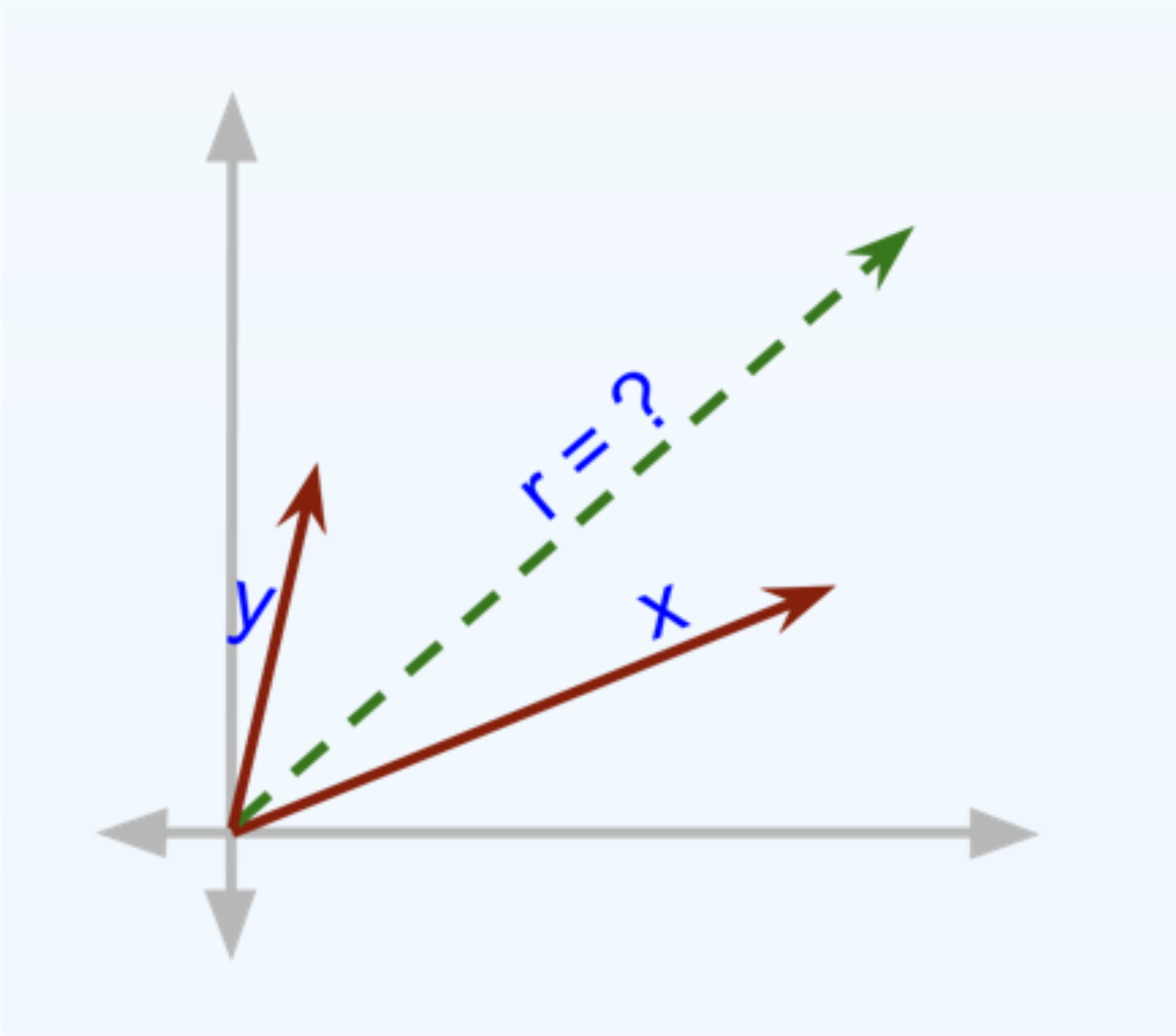
# Review of the Pythagorean theorem

If $\vec{x} = (x_1, x_2)$ and $\vec{y} = (y_1, y_2)$, define

$$\|\vec{x}\| = \left(|x_1|^2 + |x_2|^2\right)^{1/2}$$

$$\|\vec{y}\| = \left(|y_1|^2 + |y_2|^2\right)^{1/2}$$

$$\langle \vec{x}, \vec{y} \rangle = x_1\overline{y_1} + x_2\overline{y_2}.$$



# Review of the Pythagorean theorem

We say that $\vec{x}$ is orthogonal to $\vec{y}$, and write $\vec{x} \perp \vec{y}$, if $\langle \vec{x}, \vec{y} \rangle = 0$.

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**Theorem**: If $\vec{x} \perp \vec{y}$, then

$$||\vec{x} + \vec{y}||^2 = ||\vec{x}||^2 + ||\vec{y}||^2.$$

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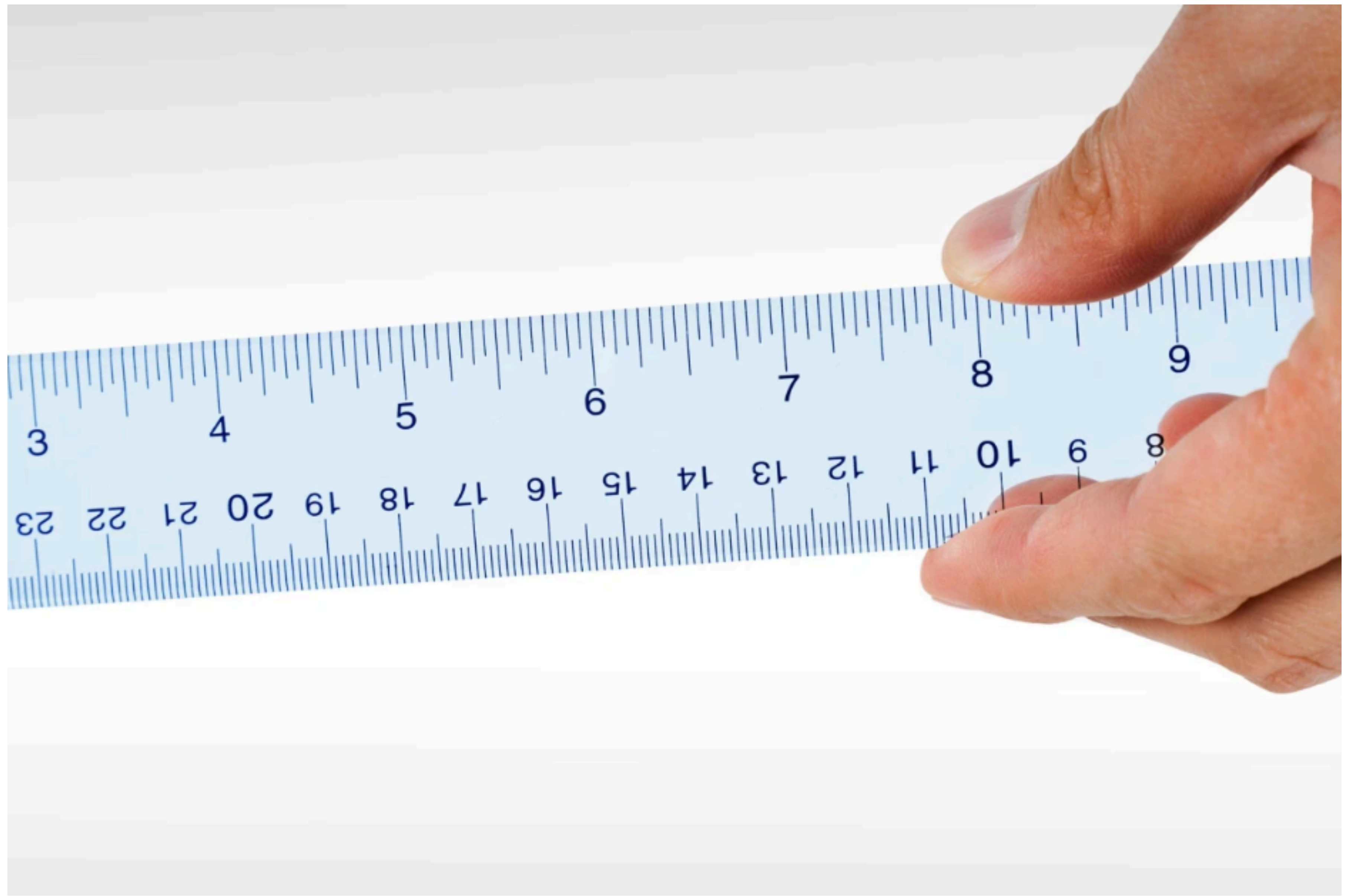

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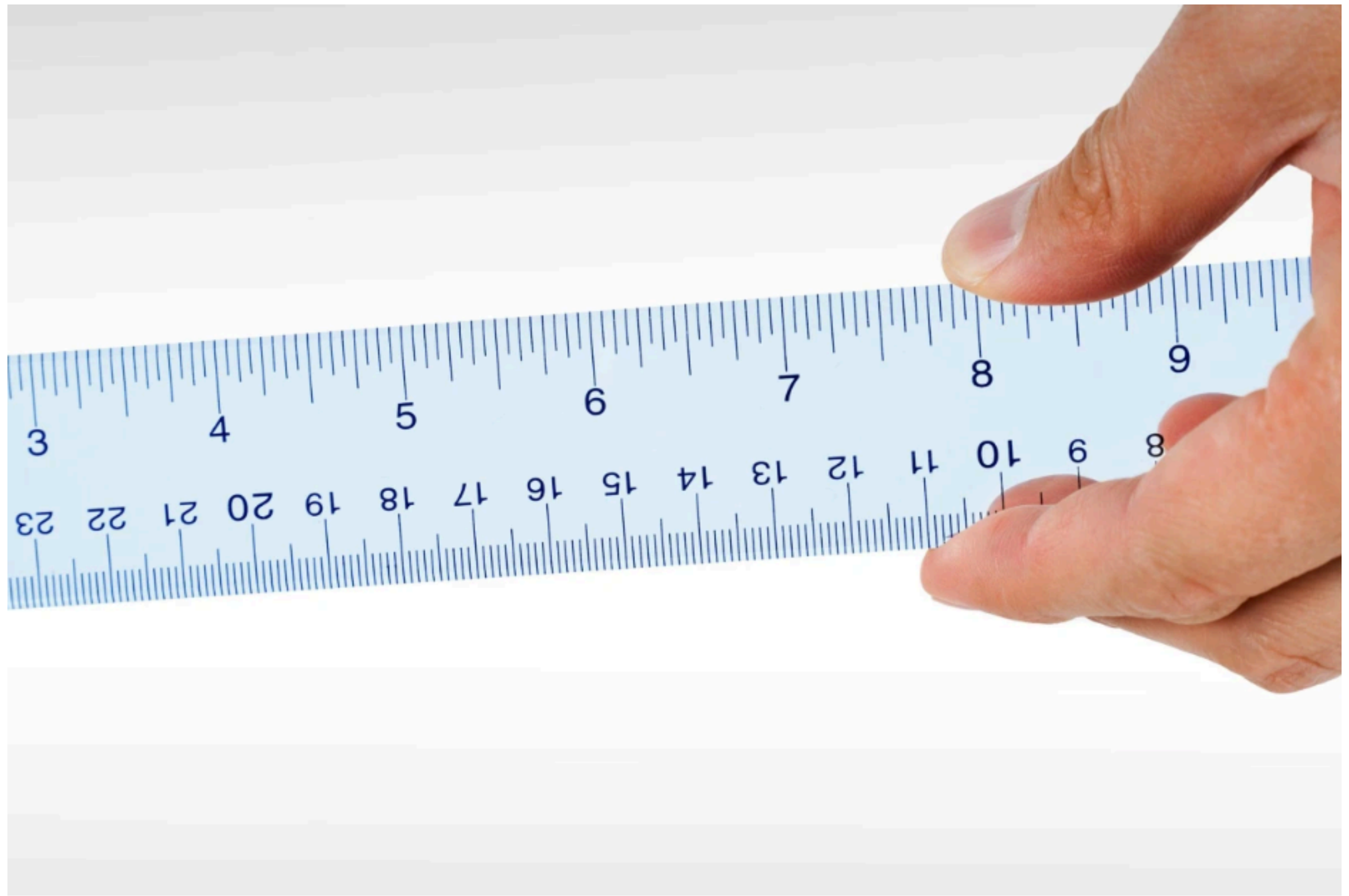
A norm on a vector space is a notion of "length."

**Example**: Let $1 < p < \infty$. Define

$$\|\vec{x}\|_p = \left( |x_1|^p + |x_2|^p \right)^{1/p}$$

for all $\vec{x} = (x_1, x_2)$.



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**Example**: Let $1 < p < \infty$. Define

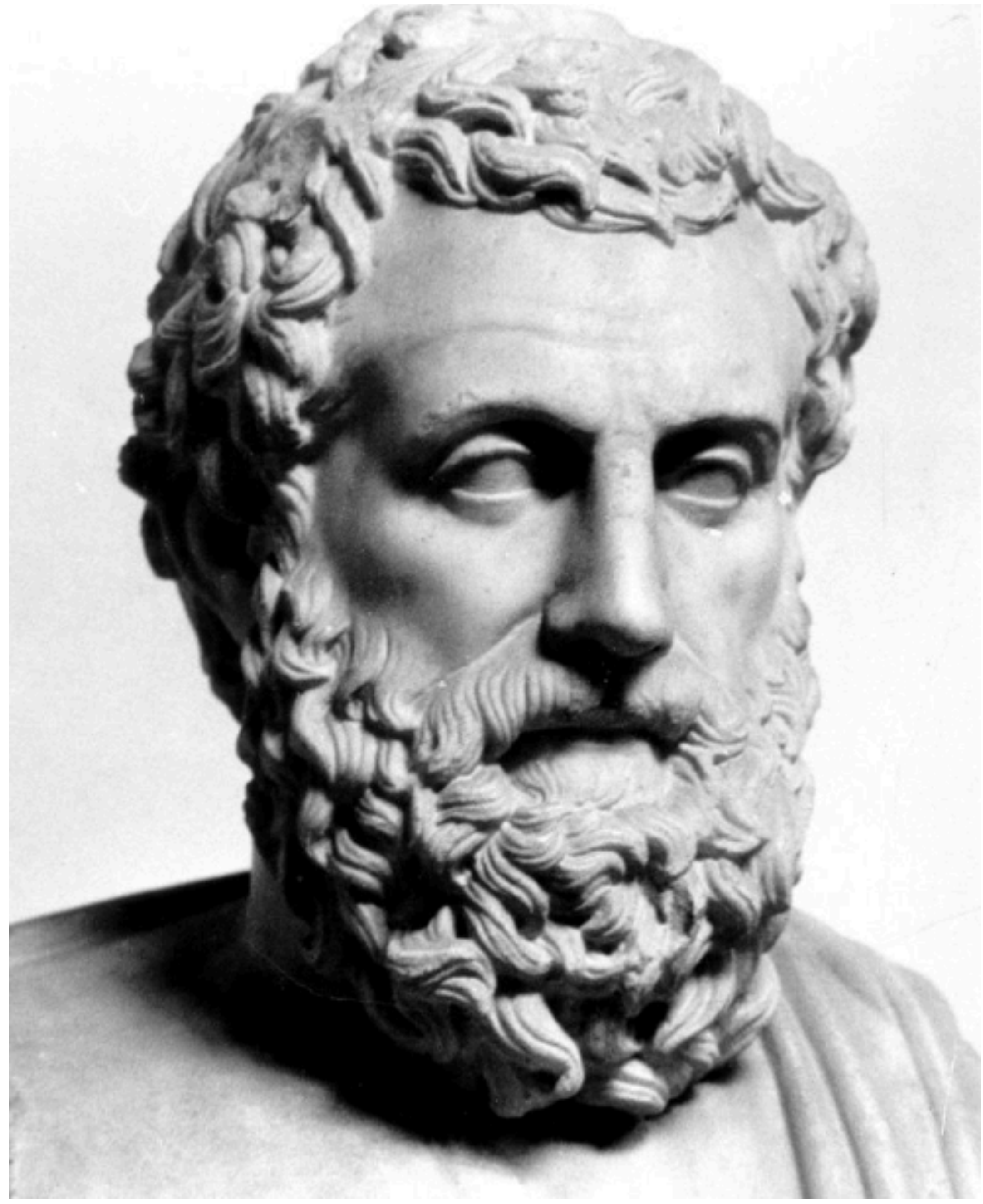
$$\|f\|_p = \left( \int_X |f(x)|^p \, d\mu \right)^{1/p}$$

for all $f \in L^p(X, \mu)$.

# Orthogonality in normed spaces

Isosceles orthogonality:

$$\vec{x} \perp_I \vec{y} \quad \text{if} \quad \|\vec{x}+\vec{y}\| = \|\vec{x}-\vec{y}\|.$$



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Roberts orthogonality

$$\vec{x} \perp_R \vec{y} \quad \text{if} \quad \|\vec{x}+c\vec{y}\| = \|\vec{x}-c\vec{y}\| \quad \text{for all } c.$$

# Orthogonality in normed spaces

Birkhoff-James orthogonality:

$$\vec{x} \perp_B \vec{y} \quad \text{if} \quad ||\vec{x} + c\vec{y}|| \geq ||\vec{x}|| \quad \text{for all } c.$$

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For $1 < p < \infty$,

$$(x_1, x_2) \perp_B (y_1, y_2) \quad \text{iff} \quad |x_1|^{p-2}\overline{x_1}y_1 + |x_2|^{p-2}\overline{x_2}y_2 = 0.$$

# Extension of the Pythagorean theorem

**Theorem** (W. Ross and YHP, 2015)

Let $1 < p \leq 2$. If $\vec{x} \perp_B \vec{y}$, then

$$\|\vec{x}\|_p^p + (2^{p-1} - 1)^{-1}\|\vec{y}\|_p^p \geq \|\vec{x} + \vec{y}\|_p^p$$

$$\|\vec{x}\|_p^2 + (p - 1)\|\vec{y}\|_p^2 \leq \|\vec{x} + \vec{y}\|_p^2$$

The inequalities reverse when $2 \leq p < \infty$.

# Applications

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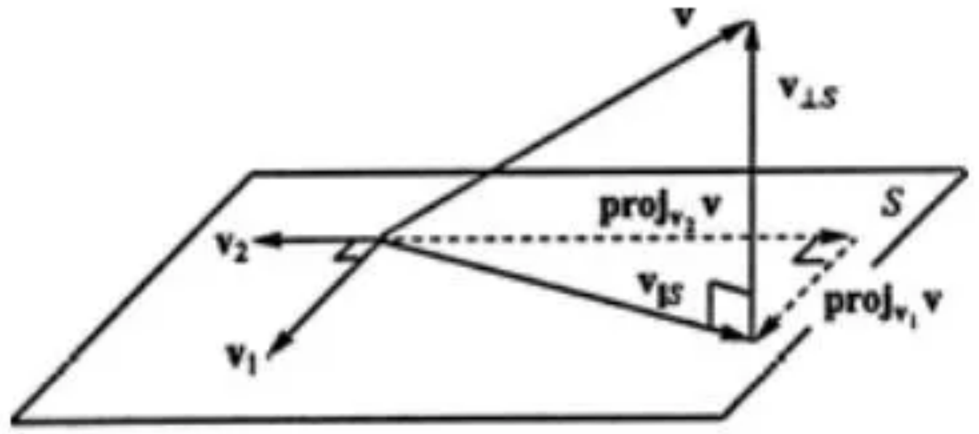
Let $\mathscr{M}$ be a subspace of a normed space $\mathscr{X}$. For any $\vec{x} \in \mathscr{X}$,
the metric projection of $\vec{x}$ onto $\mathscr{M}$ is the vector $\vec{y} \in \mathscr{M}$ satisfying

$$\|\vec{x} - \vec{y}\| \leq \|\vec{x} - \vec{z}\| \text{ for all } \vec{z} \in \mathscr{M}.$$




We write $P_{\mathscr{M}}\vec{x} = \vec{y}$.
(If $\mathscr{X}$ is complete and uniformly convex, then $\vec{y}$ exists and is unique.)

# Applications

**Theorem** (J. Mashreghi, W. Ross and YHP, 2019)

Let $\mathscr{M}_1 \subseteq \mathscr{M}_2 \subseteq \mathscr{M}_3 \subseteq \cdots$ and $\mathscr{M}_\infty = \overline{\bigcup_{n=1}^{\infty} \mathscr{M}_n}$ be subspaces of $L^p$, where $1 < p < \infty$. If $P_n$ is the metric projection onto $\mathscr{M}_n$, then

$$\lim_{n \to \infty} P_n f = P_\infty f \quad \text{for all } f \in L^p.$$

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$$\lim_{n\to\infty} P_n f = P_\infty f \quad \text{for all } f \in L^p.$$

**Proof**: If $m < n$, then $(f - P_n f) \perp_B (P_m f - P_n f)$

$$\|f - P_m f\|_p^r \geq \|f - P_n f\|_p^r + K\|P_m f - P_n f\|_p^r$$

## Applications

Let $1 < p < \infty$. Consider the set of analytic functions $f(z)$ on the open unit disk $\mathbb{D}$ such that

$$\|f\|_p = \Big( \sum_{n=0}^{\infty} |\hat{f}_n|^p \Big)^{1/p} < \infty.$$

Characterize the zero sets of such $f(z)$.

(Dragas & YHP, 2018; Mashreghi, Ross & YHP 2019; YHP, 2019).

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Define $Q_n(z) = (1 - z/w_1)(1 - z/w_2)\cdots(1 - z/w_n)$.

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Let $J_n$ be $Q_n$ minus its metric projection onto the subspace spanned by $zQ_n(z), z^2 Q_n(z), z^3 Q_n(z), \ldots$.

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Then $W$ is a zero set for some nontrivial $f$ iff $\sup_n \|J_n\|_p < \infty$.

## Applications

Let $p > 2$ be an even integer. Consider the set of analytic functions $f(z)$ on the open unit disk such that

$$\|f\|_p = \lim_{r \to 1-} \left( \int_0^{2\pi} |f(re^{i\theta})|^p \frac{d\theta}{2\pi} \right)^{1/p} < \infty.$$

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Find the polynomial $Q(z)$ of degree $d$ minimizing $\|1 - Q(z)f(z)\|_p$.

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**Theorem** (C. Felder and YHP, 2021)
Let $p > 2$ be an even integer. The roots of the polynomial $Q(z)$ of degree $d$ minimizing $||1 - Q(z)f(z)||_p$ are bounded away from the origin by a distance depending only on $p$.

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**Proof**: Apply the Pythagorean inequalities 17 times!

# The End