History of voting theory

The linear algebra of voting theory

Positional voting

Independence of irrelevant alternatives

Group actions on profiles

The geometry of positional voting methods

The geometry of IIA

The geometry of linear voting methods

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Table of contents

1. History of voting theory
2. The linear algebra of voting theory
3. Positional voting
4. Independence of irrelevant alternatives
5. Group actions on profiles
6. The geometry of positional voting methods
7. The geometry of IIA
8. Other results

History

Jean-Charles de Borda, 1733 - 1799

- French mathematician, physicist, political scientist
- Formulated a ranked preferential voting system - the *Borda Count*

Marquis de Condorcet (1743 - 1794)

- French philosopher, mathematician, and political scientist
- Formulated the *Condorcet method* of determining the winner of an election

Kenneth Arrow (1921 -)

- American economist (neo-classical economic theory)
- Arrow's impossibility theorem (1951)
- Nobel prize in economics

Modern history

Donald Saari (UC Irvine)

- Geometric voting theory
- *Basic geometry of voting*, Springer, 1995.
- *Chaotic Elections! A mathematician looks at voting*, American Mathematical Society, 2001.

Michael Orrison

- Algebraic voting theory
- *Voting, the symmetric group, and representation theory* (2009), with Zajj Daugherty, Alexander K. Eustis, Gregory Minton.
- *Representation Theory of the Symmetric Group in Voting Theory and Game Theory* (2015), with Karl-Dieter Crisman

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History of voting theory

The linear algebra of voting theory

Positional voting

Independence of irrelevant alternatives

Group actions on profiles

The geometry of positional voting methods

The geometry of IIA

The mathematics of voting theory uses tools from many disciplines, including algebra, geometry, combinatorics, representation theory, and topology.

The vector space of profiles

Let $\mathbf{C} = \{C_1, \dots, C_n\}$ be the set of candidates, and let $\lambda = (\lambda_1, \dots, \lambda_m)$ be a composition of n: $\lambda_1 + \cdots + \lambda_m = n$.

Definition

A **ballot** is a λ–shaped tabloid obtained by labeling the corresponding Young diagram with candidates.

Example: Let $\mathbf{C} = \{A, B, C, D, E, F\}$, and $\lambda = (2, 3, 1)$.

$$\begin{array}{|cc|} \hline B & D \\ \hline \end{array}\hspace{-4pt}\begin{array}{l} \\ \end{array}$$

$$\begin{array}{|ccc|}\hline B & D & \\ \hline A & C & E \\ \hline F & & \\ \hline\end{array}\;, \quad \begin{array}{|ccc|}\hline A & C & \\ \hline D & B & F \\ \hline E & & \\ \hline\end{array}\;, \quad \begin{array}{|ccc|}\hline A & C & \\ \hline F & D & B \\ \hline E & & \\ \hline\end{array}\;, \ \dots$$

Let $\mathbf{C}_\lambda =$ collection of all ballots; then

$$|\mathbf{C}_\lambda| = \frac{n!}{\lambda_1! \cdots \lambda_m!}$$

The vector space of profiles

Definition

A **profile** is an $\mathbb{R}$–linear combination of ballots. We denote the vector space of all such linear combinations by P_λ:

$$P_\lambda = \bigoplus_{X \in \mathbf{C}_\lambda} \mathbb{R}X.$$

1

Question: given a profile $\mathbf{p}$, who's the winner!?

History of voting theory

The linear algebra of voting theory

Positional voting

Independence of irrelevant alternatives

Group actions on profiles

The geometry of positional voting methods

The geometry of IIA

Who's the winner?

$$\mathbf{p} = 5 \begin{array}{|c|}\hline A \\ \hline B \\ \hline C \\ \hline \end{array} + 12 \begin{array}{|c|}\hline A \\ \hline C \\ \hline B \\ \hline \end{array} + 1 \begin{array}{|c|}\hline C \\ \hline B \\ \hline A \\ \hline \end{array}$$

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History of voting theory

The linear algebra of voting theory

Positional voting

Independence of irrelevant alternatives

Group actions on profiles

The geometry of positional voting methods

The geometry of IIA

Who's the winner?

$$\mathbf{p} = 5\begin{array}{|c|}\hline A \\ \hline B \\ \hline C \\ \hline \end{array} + 5\begin{array}{|c|}\hline C \\ \hline B \\ \hline A \\ \hline \end{array} + 2\begin{array}{|c|}\hline B \\ \hline A \\ \hline C \\ \hline \end{array}$$

Who's the winner?

$$\mathbf{p} = 1 \begin{array}{|c|}\hline \text{A} \\ \hline \text{B} \\ \hline \text{C} \\ \hline \end{array} + 1 \begin{array}{|c|}\hline \text{C} \\ \hline \text{A} \\ \hline \text{B} \\ \hline \end{array} + 1 \begin{array}{|c|}\hline \text{B} \\ \hline \text{C} \\ \hline \text{A} \\ \hline \end{array}$$

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History of voting theory

The linear algebra of voting theory

Positional voting

Independence of irrelevant alternatives

Group actions on profiles

The geometry of positional voting methods

The geometry of IIA

Head-to-head races

If $\mathbf{p} = 5 \begin{array}{|c|}\hline A \\ \hline B \\ \hline C \\ \hline \end{array} + 5 \begin{array}{|c|}\hline C \\ \hline B \\ \hline A \\ \hline \end{array} + 2 \begin{array}{|c|}\hline B \\ \hline A \\ \hline C \\ \hline \end{array}$,

Head-to-head races

If $\mathbf{p} = 5\,\begin{array}{|c|}\hline A \\ \hline B \\ \hline C \\ \hline \end{array} + 5\,\begin{array}{|c|}\hline C \\ \hline B \\ \hline A \\ \hline \end{array} + 2\,\begin{array}{|c|}\hline B \\ \hline A \\ \hline C \\ \hline \end{array}$,

$$A \text{ vs. } C?$$

To determine the aggregate preference of A versus C in a profile $\mathbf{p}$, we can eliminate all other candidates and choose the majority candidate in the resulting reduced profile:

$$5\,\begin{array}{|c|}\hline A \\ \hline B \\ \hline C \\ \hline \end{array} + 5\,\begin{array}{|c|}\hline C \\ \hline B \\ \hline A \\ \hline \end{array} + 2\,\begin{array}{|c|}\hline B \\ \hline A \\ \hline C \\ \hline \end{array} \longrightarrow 7\,\begin{array}{|c|}\hline A \\ \hline C \\ \hline \end{array} + 5\,\begin{array}{|c|}\hline C \\ \hline A \\ \hline \end{array}$$

Head-to-head races

If $\mathbf{p} = \begin{array}{|c|}\hline A \\ \hline B \\ \hline C \\ \hline\end{array} + \begin{array}{|c|}\hline C \\ \hline A \\ \hline B \\ \hline\end{array} + \begin{array}{|c|}\hline B \\ \hline C \\ \hline A \\ \hline\end{array}$,

$$A \text{ vs.} B?$$

$$B \text{ vs.} C?$$

$$C \text{ vs.} A?$$

Methods of determining a winner

Definition

A **social choice function** is a map

$$F : P_\lambda \to \mathcal{P}(\mathbf{C}) \text{ (the power set of } \mathbf{C})$$

$$F(\text{ a profile }) = \{\text{ a (sub)set of winners }\}.$$

Definition

A **cardinal welfare function** is a map

$$F : P_\lambda \to \mathbb{R}^{\mathbf{C}} \text{ (ordered } \mathbf{C} - \text{tuples of numbers)}.$$

For a profile $\mathbf{p} \in P_\lambda$, we will write

$$F(\mathbf{p}) = (F(\mathbf{p})(C_1),\ F(\mathbf{p})(C_2),\ \ldots,\ F(\mathbf{p})(C_n)).$$

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History of voting theory

The linear algebra of voting theory

Positional voting

Independence of irrelevant alternatives

Group actions on profiles

The geometry of positional voting methods

The geometry of IIA

Methods of determining a winner

$$F : P_\lambda \to \mathcal{P}(\mathbf{C})$$

Examples:

- ~~Majority~~
- Plurality: Winner = candidate with most 1st-place votes

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- Plurality: Winner = candidate with most 1st-place votes
- Copeland: Winner = candidate with most head-to-head victories
- Borda Count: Winner = candidate with greatest Borda point total

Methods of determining a winner

$$F : P_\lambda \to \mathcal{P}(\mathbf{C})$$

Examples:

- ~~Majority~~
- Plurality: Winner = candidate with most 1st-place votes
- Copeland: Winner = candidate with most head-to-head victories
- Borda Count: Winner = candidate with greatest Borda point total
- Positional voting: Winner = candidate with greatest positional point total

The moral:

Linear algebra provides a natural setting to study many voting methods. Profiles form a vector space, and certain voting methods are linear transformations.

Positional voting

Let $\mathbf{C} = \{A, B, C, D, E, F\}$, and $\lambda = (2, 3, 1)$. Then

$$P_\lambda = \mathrm{sp}_{\mathbb{R}} \left\{ \begin{array}{|c|c|c|} \hline C1 & C2 & \multicolumn{1}{c}{} \\ \hline C3 & C4 & C5 \\ \hline C6 & \multicolumn{2}{c}{} \\ \cline{1-1} \end{array} : C_i \in \mathbf{C} \right\}.$$

Let $\mathbf{w} = (5, 3, 1)$. We can use these weights to define a positional social choice function $B_{\mathbf{w}}$. Let X be a candidate. Award points to X, one ballot at a time, as follows:

Each	1^{st}–place ballot position	$\longrightarrow$	5 points
	2^{nd}–place ballot position	$\longrightarrow$	3 points
	3^{rd}–place ballot position	$\longrightarrow$	1 point.

Positional voting

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	3^{rd}–place ballot position	$\longrightarrow$	1 point.

Each candidate receives a **w**–*positional point total*.

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History of voting theory

The linear algebra of voting theory

Positional voting

Independence of irrelevant alternatives

Group actions on profiles

The geometry of positional voting methods

The geometry of IIA

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Suppose . . .

$$\mathbf{p} = 2\begin{array}{|ccc|}\hline C & D & \\ \hline A & B & E \\ \hline F & & \\ \hline\end{array} + 4\begin{array}{|ccc|}\hline A & B & \\ \hline D & C & F \\ \hline E & & \\ \hline\end{array} + 7\begin{array}{|ccc|}\hline A & B & \\ \hline F & D & C \\ \hline E & & \\ \hline\end{array}.$$

Positional point totals are then . . .

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History of voting theory

The linear algebra of voting theory

Positional voting

Independence of irrelevant alternatives

Group actions on profiles

The geometry of positional voting methods

The geometry of IIA

Positional voting

Suppose ...

$$\mathbf{p} = 2\begin{array}{|ccc|}\hline C & D & \\ \hline A & B & E \\ \hline F & & \\ \hline\end{array} + 4\begin{array}{|ccc|}\hline A & B & \\ \hline D & C & F \\ \hline E & & \\ \hline\end{array} + 7\begin{array}{|ccc|}\hline A & B & \\ \hline F & D & C \\ \hline E & & \\ \hline\end{array}.$$

Positional point totals are then ...

A	$2 \cdot 3 + 4 \cdot 5 + 7 \cdot 5 = 61.$
B	$2 \cdot 3 + 4 \cdot 5 + 7 \cdot 5 = 61.$
C	$2 \cdot 5 + 4 \cdot 3 + 7 \cdot 3 = 43.$

History of voting theory

The linear algebra of voting theory

Positional voting

Independence of irrelevant alternatives

Group actions on profiles

The geometry of positional voting methods

The geometry of IIA

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Suppose . . .

$$\mathbf{p} = 2\,\begin{array}{|c|c|c|}\hline C & D & \\ \hline A & B & E \\ \hline F & & \\ \hline\end{array} + 4\,\begin{array}{|c|c|c|}\hline A & B & \\ \hline D & C & F \\ \hline E & & \\ \hline\end{array} + 7\,\begin{array}{|c|c|c|}\hline A & B & \\ \hline F & D & C \\ \hline E & & \\ \hline\end{array}.$$

Positional point totals are then . . .

A	$2 \cdot 3 + 4 \cdot 5 + 7 \cdot 5 = 61.$
B	$2 \cdot 3 + 4 \cdot 5 + 7 \cdot 5 = 61.$
C	$2 \cdot 5 + 4 \cdot 3 + 7 \cdot 3 = 43.$
D	$2 \cdot 5 + 4 \cdot 3 + 7 \cdot 3 = 43.$
E	$2 \cdot 3 + 4 \cdot 1 + 7 \cdot 1 = 17.$
F	$2 \cdot 1 + 4 \cdot 3 + 7 \cdot 3 = 35.$

So . . .

$$B_{\mathbf{w}}(\mathbf{p}) = (\mathbf{61}, \mathbf{61}, 43, 43, 17).$$

Positional voting

- A positional voting method $B_{\mathbf{w}}$ is uniquely determined by the weight vector $\mathbf{w} = (w_1, \ldots, w_m)$.
- Typically one requires $w_i \geq w_{i+1}$, and $w_1 > w_m$.
- $B_{\mathbf{w}}$ is a linear social choice function.
- When $\lambda = (1, 1, \ldots, 1)$ and $\mathbf{w} = (n, n-1, \ldots, 2, 1)$, $B_{\mathbf{w}}$ is the *Borda Count Method*: used for some parlimentiary and presidential elections, NBA MVP, ...
- When $\lambda = (1, 1, \ldots, 1)$ and $\mathbf{w} = (1, 0, \ldots, 0)$, $B_{\mathbf{w}}$ is the *Plurality Method*.

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History of voting theory

The linear algebra of voting theory

Positional voting

Independence of irrelevant alternatives

Group actions on profiles

The geometry of positional voting methods

The geometry of IIA

Problems with positional voting

- Majority candidates are not guaranteed victory. 2
- Candidates who win all head–to–head races are not guaranteed victory.
- Irrelevant candidates can sometimes manipulate the outcome of the election. Example...

3

Independence of irrelevant alternatives

Definition

Let $X, Y \in \mathbf{C}$. We say two profiles $\mathbf{p}$, $\mathbf{q}$ are XY**–equivalent**, written

$$\mathbf{p} \sim_{XY} \mathbf{q},$$

if all voters ranking X above Y in $\mathbf{p}$ also rank X above Y in $\mathbf{q}$, and vice versa.

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History of voting theory
The linear algebra of voting theory
Positional voting
Independence of irrelevant alternatives
Group actions on profiles
The geometry of positional voting methods
The geometry of IIA

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4

Remarks:

- $\sim_{XY}$ is an equivalence relation
- If $\mathbf{p} \sim_{XY} \mathbf{q}$, and X defeats Y in a head–to–head race in $\mathbf{p}$, then X does so in $\mathbf{q}$ as well.

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Remarks:

- $\sim_{XY}$ is an equivalence relation
- If $\mathbf{p} \sim_{XY} \mathbf{q}$, and X defeats Y in a head–to–head race in $\mathbf{p}$, then X does so in $\mathbf{q}$ as well.

Independence of Irrelevant Alternatives

Definition

We say that a cardinal welfare function F satisfies the **IIA criterion** if, given any two XY–equivalent profiles $\mathbf{p}$, $\mathbf{q}$,

$$F(\mathbf{p})(X) > F(\mathbf{p})(Y) \iff F(\mathbf{q})(X) > F(\mathbf{q})(Y).$$

The Borda Count method *violates* this criterion; it is susceptible to *insincere voting*.

Group actions on the profile space

There are (at least) two ways that we can transform profiles:

1. **Discrete transformations**:

$P_{\mathbf{C}}$ = Permutations of the set of candidates;
acts (naturally) upon $\mathbb{R}^{\mathbf{C}}$ by permuting candidates

Group actions on the profile space

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1. **Discrete transformations**:

   $P_{\mathbf{C}}$ = Permutations of the set of candidates;
   acts (naturally) upon $\mathbb{R}^{\mathbf{C}}$ by permuting candidates
   acts upon P^{λ} via permutation matrices

2. **Continuous transformations**:

   $SO(P^{\lambda})$ = (generalized) rotations of the profile space P^{λ}

Group actions on the profile space

The moral:

One of the (many) advantages of constructing a vector space of profiles is that we can allow groups to act on these profiles. This, then, falls under the subject of (group) representation theory.

Positional voting via inner products

Proposition

Let $\lambda = (\lambda_1, \ldots, \lambda_m)$ be a composition of n, and let $\mathbf{w} = (w_1, \ldots, w_m)$. Let X and Y be distinct candidates, and $\mathbf{p}$, $\mathbf{q}$ be two profiles.

1. *There exists a vector $\mathbf{r}_{XY}$ such that $\mathbf{p} \sim_{XY} \mathbf{q}$ if and only if*

$$(\mathbf{p} - \mathbf{q}) \cdot \mathbf{r}_{XY} = 0.$$

2. *There exists a vector $\mathbf{v}_X$ such that*

$$B_{\mathbf{w}}(\mathbf{p})(X) = \mathbf{p} \cdot \mathbf{v}_X.$$

Positional voting via inner products

Let $B_{\mathbf{w}}$ be a positional voting method, and let X and Y be two candidates. For which profiles $\mathbf{p}$ does $B_{\mathbf{w}}(\mathbf{p})$ rank X above Y?

Corollary

Let $\mathbf{v}_{XY} = \mathbf{v}_X - \mathbf{v}_Y$. When using the positional method $B_{\mathbf{w}}$ evaluated at a profile $\mathbf{p}$, candidate X defeats Y if and only if $\mathbf{p} \cdot \mathbf{v}_{XY} > 0$.
So the collection of all profiles $\mathbf{p}$ such that $B_{\mathbf{w}}(\mathbf{p})$ ranks X above Y consists of the 'positive half-space'

$$(\mathbf{v}_{XY})_+ = \{\mathbf{p} : \mathbf{p} \cdot (\mathbf{v}_{X>Y}) > 0\}.$$

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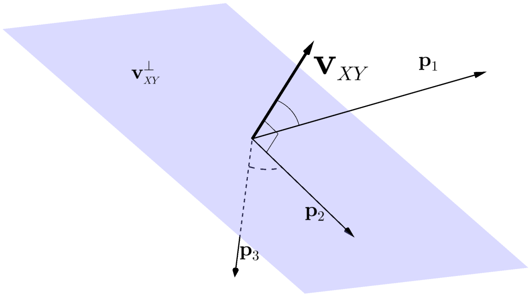
History of voting theory
The linear algebra of voting theory
Positional voting
Independence of irrelevant alternatives
Group actions on profiles
The geometry of positional voting methods
The geometry of IIA

Positional voting via inner products



Rotations about $\mathbf{r}_{XY}$ preserve XY–equivalence

Theorem

If $\mathbf{p} \sim_{XY} \mathbf{q}$, there exists a rotation about $\mathbf{r}_{XY}$ which rotates $\mathbf{p}$ to $\mathbf{q}$.

Conversely...

Theorem

If T is a rotation about $\mathbf{r}_{XY}$, then $\mathbf{p} \sim_{XY} T(\mathbf{p})$.

The geometry of IIA

Suppose F satisfies IIA. Then, for $\mathbf{p} \sim_{XY} \mathbf{q}$,

if X defeats Y in $F(\mathbf{p})$, then X defeats Y in $F(\mathbf{q})$.

The geometry of IIA

Suppose F satisfies IIA. Then, for $\mathbf{p} \sim_{XY} \mathbf{q}$,

$$F(\mathbf{p})(X) > F(\mathbf{p})(Y) \quad \Rightarrow \quad F(\mathbf{q})(X) > F(\mathbf{q})(Y)$$

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History of voting theory

The linear algebra of voting theory

Positional voting

Independence of irrelevant alternatives

Group actions on profiles

The geometry of positional voting methods

The geometry of IIA

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Suppose F satisfies IIA. Then, for $\mathbf{p} \sim_{XY} \mathbf{q}$,

$$\mathbf{p} \cdot \mathbf{v}_{XY} > 0 \;\Rightarrow\; \mathbf{q} \cdot \mathbf{v}_{XY} > 0.$$

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Tim Ridenour
Baruch College

History of voting theory

The linear algebra of voting theory

Positional voting

Independence of irrelevant alternatives

Group actions on profiles

The geometry of positional voting methods

The geometry of IIA

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Suppose F satisfies IIA. Then, for $\mathbf{p} \sim_{XY} \mathbf{q}$,

$$\mathbf{p} \in (\mathbf{v}_{XY})_+ \ \Rightarrow \ \mathbf{q} \in (\mathbf{v}_{XY})_+.$$

The geometry of IIA

Suppose F satisfies IIA. Then, for any $\mathbf{p} \in P^\lambda$,

$$\mathbf{p} \in (\mathbf{v}_{XY})_+ \;\; \Rightarrow \;\; T(\mathbf{p}) \in (\mathbf{v}_{XY})_+,$$

for any rotation T about $\mathbf{r}_{XY}$.

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History of voting theory

The linear algebra of voting theory

Positional voting

Independence of irrelevant alternatives

Group actions on profiles

The geometry of positional voting methods

The geometry of IIA

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Suppose F satisfies IIA. Then

$$SO(P^{\lambda})^{\mathbf{r}_{XY}}\left((\mathbf{v}_{XY})_{+}\right) \subseteq (\mathbf{v}_{XY})_{+}.$$

The geometry of IIA

Suppose F satisfies IIA. Then ...

Rotation about $\mathbf{r}_{XY}$ preserves the line $\mathbb{R}\mathbf{v}_{XY}$.

But rotation about $\mathbf{r}_{XY}$ preserves $\mathbb{R}\mathbf{v}_{XY}$ if and only if $\mathbf{r}_{XY} \parallel \mathbf{v}_{XY}$.

Moral: To determine if a particular positional voting method $B_{\mathbf{w}}$ satisfies IIA, we only need to calculate the vectors $\mathbf{v}_{XY}$, $\mathbf{r}_{XY}$ and check:

$$\text{is} \quad \mathbf{r}_{XY} \parallel \mathbf{v}_{XY} \ ?$$

The failure of $B_{\mathbf{w}}$

Theorem (Ridenour, S)

Let $|\mathbf{C}| \geq 3$ and $|\lambda| > 2$. Then all positional voting methods violate IIA.

Proof.

The vectors $\mathbf{v}_{XY}$ and $\mathbf{r}_{XY}$ are *never* parallel. □

Strong majority

Definition

A CWF F satisfies the *strong majority criterion* if, whenever a candidate X defeats a candidate Y in a head–to–head race in a profile $\mathbf{p}$, we have $F(\mathbf{p})(X) > F(\mathbf{p})(Y)$; i.e., X defeats Y in the election.

Proposition

The positional voting method $B_{\mathbf{w}}^{\lambda}$ satisfies the strong majority criterion if and only if $(\mathbf{r}_{XY})_+ \subseteq (\mathbf{v}_{XY})_+$.

Theorem (Ridenour, S)

If $|\lambda| = 2$, then the positional voting method $B_{\mathbf{w}}^{\lambda}$ satisfies the strong majority criterion if and only if $\mathbf{w}(1) > \mathbf{w}(2)$. If $|\lambda| > 2$ then no nontrivial positional voting method satisfies strong majority.

Strong majority

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Proposition

The positional voting method $B_{\mathbf{w}}^{\lambda}$ satisfies the strong majority criterion if and only if $(\mathbf{r}_{XY})_+ \subseteq (\mathbf{v}_{XY})_+$.

Theorem (Ridenour, S)

If $|\lambda| = 2$, then the positional voting method $B_{\mathbf{w}}^{\lambda}$ satisfies the strong majority criterion if and only if $\mathbf{w}(1) > \mathbf{w}(2)$. If $|\lambda| > 2$ then no nontrivial positional voting method satisfies strong majority.

Pareto efficiency

Proposition

The positional voting method $B_{\mathbf{w}}^{\lambda}$ is Pareto efficient if and only if $\mathbf{w}$ is strictly decreasing; i.e., $\mathbf{w}(i) > \mathbf{w}(i+1)$.

The Condorcet criterion

Proposition

A candidate X is a Condorcet candidate in the profile $\mathbf{p}$ if and only if, given any other candidate $Y \in \mathbf{C}$, the $S^X_{\mathbf{C}}$–orbit of $\mathbf{p}$ is contained in $(\mathbf{r}_{XY})_+$. The CWF F satisfies the Condorcet criterion if and only if

$$S^X_{\mathbf{C}}.\mathbf{p} \subseteq (\mathbf{r}_{XY})_+ \;\Rightarrow\; F(S^X_{\mathbf{C}}.\mathbf{p}) \subseteq (F(\mathbf{r}_{XY}))_+. \tag{8.1}$$

If F is realized as a positional voting method $B^\lambda_{\mathbf{w}}$, then this condition is equivalent to

$$S^X_{\mathbf{C}}.\mathbf{p} \subseteq (\mathbf{r}_{XY})_+ \;\Rightarrow\; S^X_{\mathbf{C}}.\mathbf{p} \subseteq (\mathbf{v}_{XY})_+.$$

The morals:

1. **Profiles (of ballots) form a vector space, and we can 'manipulate' these profiles by allowing groups to act upon this vector space.**
2. **The behavior and results of linear voting methods are completely determined by the configurations of, and inner products with, certain families of vectors $\{\mathbf{v}_{XY}\}$ and $\{\mathbf{r}_{XY}\}$ in the vector space of profiles.**

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History of voting theory

The linear algebra of voting theory

Positional voting

Independence of irrelevant alternatives

Group actions on profiles

The geometry of positional voting methods

The geometry of IIA

Next time...

$$\{\mathbf{v}_{XY}\} \leftrightarrow \text{Lie algebras ?}$$

... don't miss it!

Thank you!

Interested? Questions? Ideas?

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