The nonnegative integers form . . . a field??

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What's this all about?

- Roughly speaking, a field is an algebraic system, such as the real numbers, where one can add, subtract, multiply, and divide (except by zero).
- Under the usual addition and multiplication of numbers, the set $N = \{0, 1, 2, \ldots\}$ of nonnegative integers is not a field.
- There is a way, using some unusual operations, to make N into a field.
- This talk is about it's done.

What to expect

- The natural numbers $\mathbb{Z}^+ \cup \{0\} = \{0, 1, 2, \dots\}$
- Nimber addition
- Nimber multiplication
- The fields $\mathcal{F}_2$ and $\mathcal{F}_4$
- The field of natural nimbers

Origins

- Nimbers and nimber arithmetic originated in the world of two-player combinatorial games – in particular, from the game called *nim*.
- Every position in a combinatorial game has a *Grundy number*.
- Suppose it's your turn to play.
  - If your position's Grundy number is nonzero, then there is a move that will transform the position into one with Grundy number 0, and with best play, you will win.
  - If your position's Grundy number is 0, then every move you make will transform the position into one with nonzero Grundy number, and with best play, your opponent will win.
- Grundy numbers are also called *nimbers*.

What's a Nimber?

A nimber is a number with a little extra finery,

You write it as a string of ones and zeros: that's in binary.

To add them, line a few strings up and then perform exclusive-or.

To multiply them takes some work, but hey! that's what this talk is for.

Nimber addition: an example

Let's add the nimbers 22, 37, and 18.

Write them in binary: $22 = 16 + 4 + 2 = 010110$, $37 = 32 + 4 + 1 = 100101$, and $18 = 16 + 2 = 010010$.

Add the strings by a string exclusive-or (XOR), written as $\oplus$:

$$\begin{aligned} 010110 &= 22 \\ 100101 &= 37 \\ 010010 &= 18 \\ 100001 &= 33 = 22 \oplus 37 \oplus 18 \end{aligned}$$

Thus, $22 \oplus 37 \oplus 18 = 33$.

Nimber addition and groups

The n-bit nimbers $\{0, 1, \ldots, 2^n - 1\}$ form a **group** under nimber addition.

Here are those groups of orders 2, 4, and 8:

$\oplus$	0	1
0	0	1
1	1	0

$\oplus$	**0**	**1**	2	3
0	**0**	**1**	2	3
1	**1**	**0**	3	2
2	2	3	0	1
3	3	2	1	0

$\oplus$	**0**	**1**	**2**	**3**	4	5	6	7
0	**0**	**1**	**2**	**3**	4	5	6	7
1	**1**	**0**	**3**	**2**	5	4	7	6
2	**2**	**3**	**0**	**1**	6	7	4	5
3	**3**	**2**	**1**	**0**	7	6	5	4
4	4	5	6	7	0	1	2	3
5	5	4	7	6	1	0	3	2
6	6	7	4	5	2	3	0	1
7	7	6	5	4	3	2	1	0

Notice that these groups are nested.

The additive group of nimbers of order 16

$\oplus$	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
0	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
1	1	0	3	2	5	4	7	6	9	8	11	10	13	12	15	14
2	2	3	0	1	6	7	4	5	10	11	8	9	14	15	12	13
3	3	2	1	0	7	6	5	4	11	10	9	8	15	14	13	12
4	4	5	6	7	0	1	2	3	12	13	14	15	8	9	10	11
5	5	4	7	6	1	0	3	2	13	12	15	14	9	8	11	10
6	6	7	4	5	2	3	0	1	14	15	12	13	10	11	8	9
7	7	6	5	4	3	2	1	0	15	14	13	12	11	10	9	8
8	8	9	10	11	12	13	14	15	0	1	2	3	4	5	6	7
9	9	8	11	10	13	12	15	14	1	0	3	2	5	4	7	6
10	10	11	8	9	14	15	12	13	2	3	0	1	6	7	4	5
11	11	10	9	8	15	14	13	12	3	2	1	0	7	6	5	4
12	12	13	14	15	8	9	10	11	4	5	6	7	0	1	2	3
13	13	12	15	14	9	8	11	10	5	4	7	6	1	0	3	2
14	14	15	12	13	10	11	8	9	6	7	4	5	2	3	0	1
15	15	14	13	12	11	10	9	8	7	6	5	4	3	2	1	0

Nimber multiplication

Nimber multiplication is more complicated than nimber addition.

Let's see how it works.

Nimber multiplication and the Fermat powers

The nimbers 2^{2^n}, where n is a nonnegative integer, are called **Fermat powers**. The first five Fermat powers are
$2 = 2^{2^0}, 4 = 2^{2^1}, 16 = 2^{2^2}, 256 = 2^{2^3}$, and $65536 = 2^{2^4}$.

Denote the **nim product** by $\otimes$ and let $\cdot$ denote the usual integer product. If k is a nonnegative integer and $k < 2^{2^n}$, then $k \otimes 2^{2^n} = k \cdot 2^{2^n}$.

Thus, $4 \otimes 3 = 4 \cdot 3 = 12$ and $16 \otimes 13 = 16 \cdot 13 = 208$.

However, $2^{2^n} \otimes 2^{2^n} = \frac{3}{2} \cdot 2^{2^n} = 3 \cdot 2^{2^n - 1}$.

Nimber multiplication: features

$0 \otimes n = 0$ and $1 \otimes n = n$ for all nimbers n;

$\otimes$ is associative and commutative:

$$a \otimes (b \otimes c) = (a \otimes b) \otimes c \text{ and } a \otimes b = b \otimes a;$$

$\otimes$ distributes over $\oplus$: $a \otimes (b \oplus c) = (a \otimes b) \oplus (a \otimes c)$; and

The Freshman's Dream: Denote $x \otimes x$ by x^2. Then

$$(a \oplus b)^2 = a^2 \oplus b^2.$$

Nimber products in $\{0, 1, 2, 3\}$

The rules for multiplying by 0 and 1 give us the partial table on the left:

$\otimes$	0	1	2	3
0	0	0	0	0
1	0	1	2	3
2	0	2		
3	0	3		

$\otimes$	0	1	2	3
0	0	0	0	0
1	0	1	2	3
2	0	2	3	1
3	0	3	1	2

As for the others:

$$\begin{aligned} 2^2 &= 3 \cdot 2^{2-1} = 3 \text{ (by definition)} \\ 2 \otimes 3 &= 2 \otimes (1 \oplus 2) = 2 \oplus 3 = 1, \text{ and} \\ 3^2 &= (1 \oplus 2)^2 = 1^2 \oplus 2^2 = 1 \oplus 3 = 2. \end{aligned}$$

The 2-element field of nimbers

The nimbers $\mathcal{F}_2 = \{0, 1\}$ are a field under the operations $\oplus$ and $\otimes$:

$\oplus$	0	1
0	0	1
1	1	0

$\otimes$	0	1
0	0	0
1	0	1

This is the same as mod-2 integer arithmetic.

The 4-element field of nimbers

The nimbers $\mathcal{F}_4 = \{0, 1, 2, 3\}$ are a field under the operations $\oplus$ and $\otimes$:

$\oplus$	0	1	2	3
0	0	1	2	3
1	1	0	3	2
2	2	3	0	1
3	3	2	1	0

$\otimes$	0	1	2	3
0	0	0	0	0
1	0	1	2	3
2	0	2	3	1
3	0	3	1	2

$\oplus$ and $\otimes$ obey the usual laws of integer arithmetic – with two exceptions:

- Every nonzero nimber has an inverse with respect to $\otimes$, and
- $n \oplus n = 0$ for all nimbers n.

Now let's look at some more nimber products.

A nimber square: $8 \otimes 8$

Rule of thumb: Express the factors as sums of powers of 2, use the Fermat powers when possible, and work the distributive law for all it is worth.

Example: 8 is not a Fermat power, but $8 = 2 \otimes 4$, and so:

$$\begin{aligned} 8^2 &= (2 \otimes 4) \otimes (2 \otimes 4) \\ &= (2 \otimes 2) \otimes (4 \otimes 4) \ldots \text{ rearrange factors} \\ &= 3 \otimes 6 \ldots \text{ squaring Fermat powers} \\ &= 3 \otimes (2 \oplus 4) \ldots \text{ using a nim sum} \\ &= (3 \otimes 2) \oplus (3 \otimes 4) = 1 \oplus 12 \\ &= 13. \end{aligned}$$

Hence, $8 \otimes 8 = 13$.

A nimber product: $7 \otimes 11$

Let's try $7 \otimes 11$. Write $7 = 3 \otimes 4, 11 = 3 \oplus 8$, distribute, use previous results:

$$\begin{aligned}
7 \otimes 11 &= (3 \oplus 4) \otimes (3 \oplus 8) \\
&= 3^2 \oplus (4 \otimes 3) \oplus (3 \otimes 8) \oplus (4 \otimes (4 \otimes 2)) \\
&= 2 \oplus 12 \oplus ((3 \otimes 2) \otimes 4) \oplus (6 \otimes 2) \\
&= 2 \oplus 12 \oplus (1 \otimes 4) \oplus (2 \oplus 4) \otimes 2 \\
&= 2 \oplus 12 \oplus 4 \oplus 3 \oplus 8 \\
&= 1 \ldots
\end{aligned}$$

. . . and so 7 and 11 are nimber multiplicative inverses.

The rest of the story

- For each Fermat power $FP(n) := 2^{2^n}$, this method constructs a field $\mathcal{F}_{2^{2^n}}$ of order $FP(n)$.
- These fields are nested. That is,

$$\mathcal{F}_2 \subseteq \mathcal{F}_4 \subseteq \mathcal{F}_{16} \subseteq \ldots \subseteq \mathcal{F}_{2^{2^n}} \subseteq \ldots$$

- Their union is a field $\mathcal{F}_\infty$ consisting of the nimbers $0, 1, 2, \ldots$
- ... and this is the field of natural nimbers.

THANK YOU!