Tracking the Mailman:

An adventure in graph theory

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MD-DC-VA Section Meeting

2013 November 2

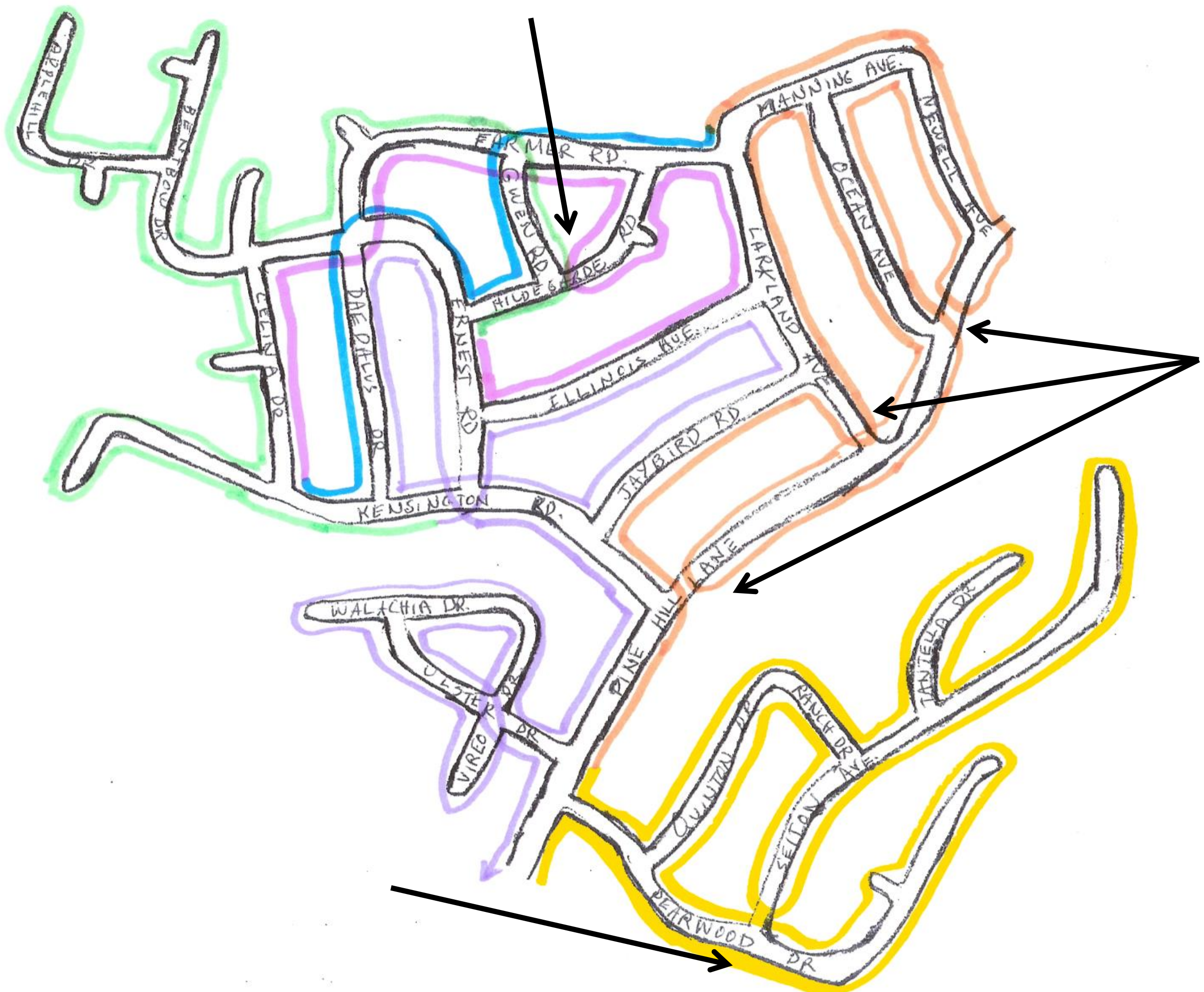
Farmville, VA

Mail Delivery

- Move in 2010
- Late mail deliveries
- Determined mail truck route
  - Several months
  - Delayed by deviations from the route

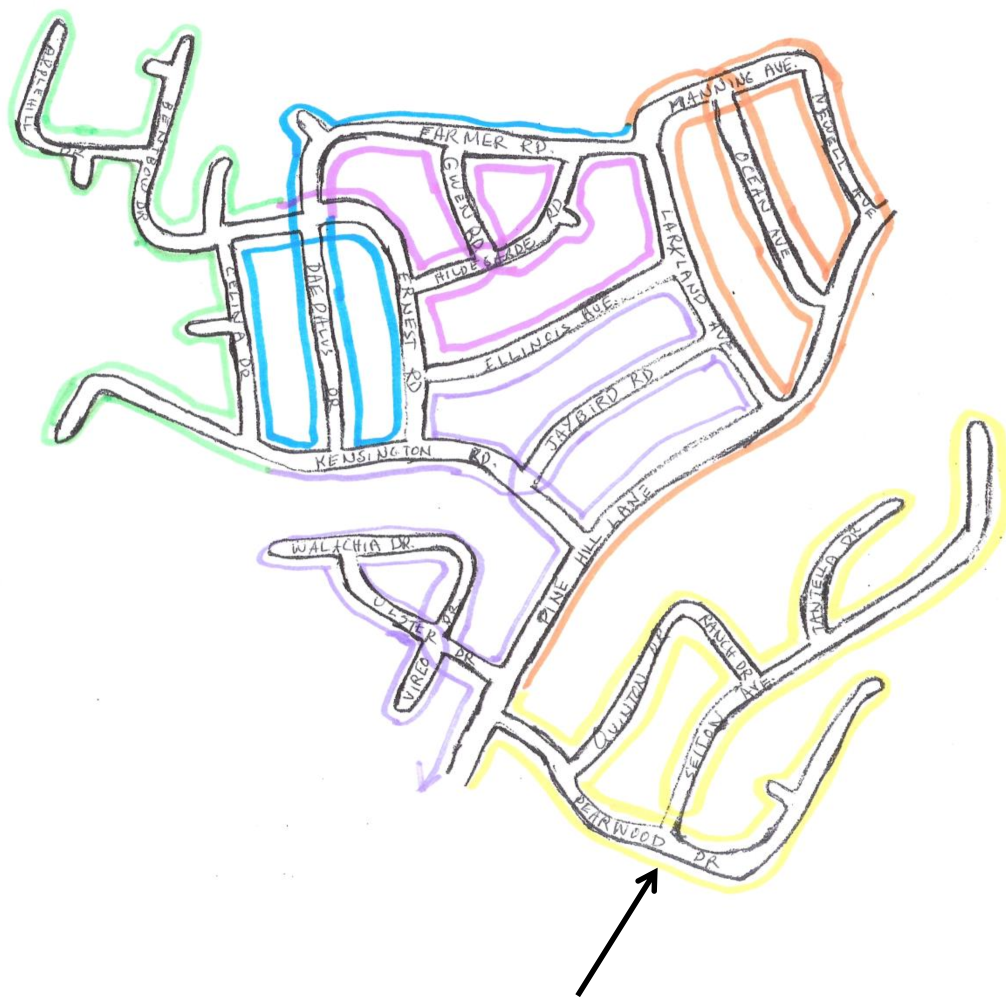
Mail Truck Route

5 U-turns



Mail Truck Route

Only 1 U-turn



Mail Route Problem

- Given a connected planar graph G, find a path on G that covers each edge exactly twice, once in each direction, and such that the two passes through each edge are not adjacent to each other (i.e., there are no U-turns).
- Exception: Cul-de-sacs

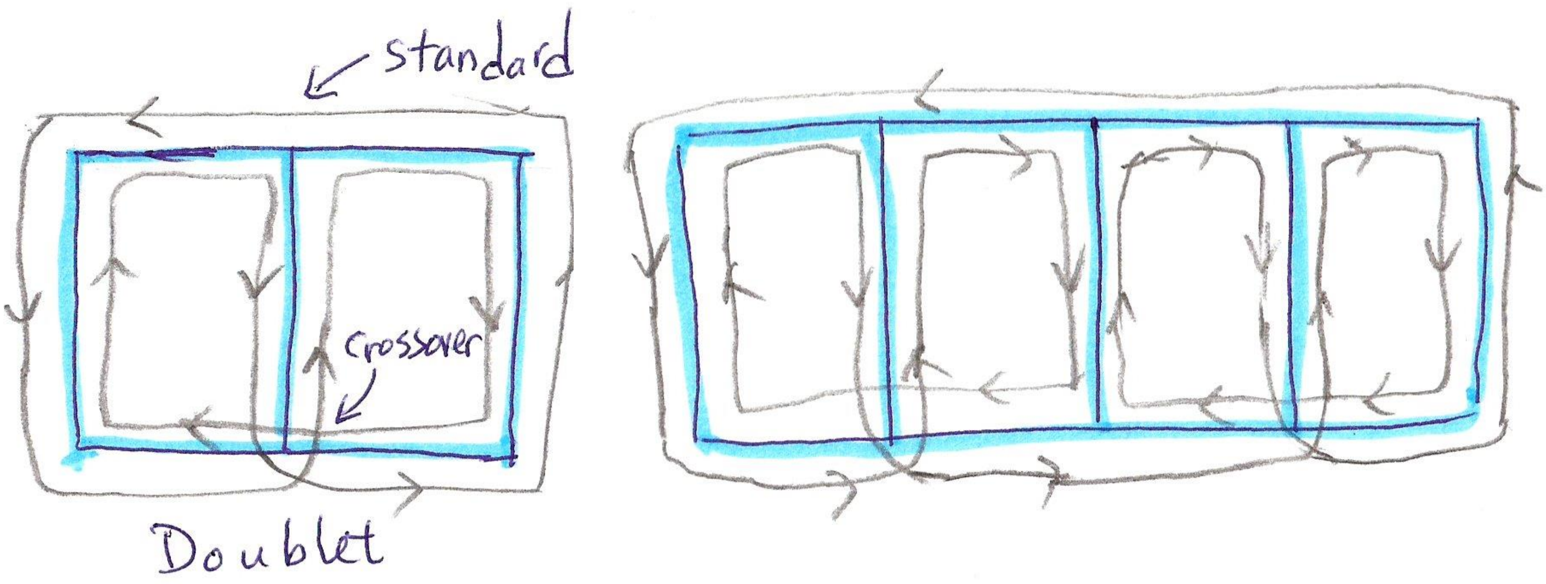
Mail Route Problem

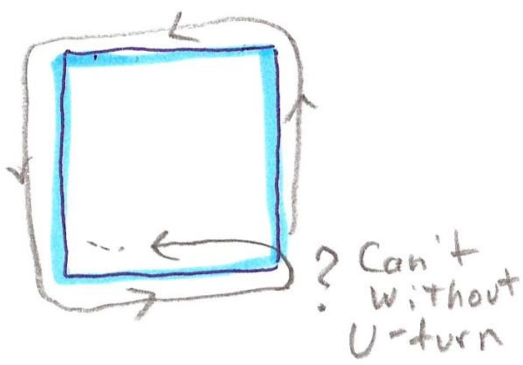
- Such a route is called *mailroutable*.
- Only planar graphs
  - Therefore, G consists of a collection of *regions*.
- No one-way streets
- Reentrant: get to the end, start over again

Eulerian Circuit

- A path that begins and ends with the same vertex and which covers each oriented edge once and only once.
- Theorem: (Hierholzer, 1873) Every connected planar graph G with indegree = outdegree for all vertices of G has an Eulerian circuit.

Examples





Mail Route Exists

- 1 x *n* strip, *n* even
- 2 x 2, 2 x 3, 3 x 3 arrays
- Octahedron (Schlegel Diagram)
- Icosahedron (Schlegel Diagram)
- *n*-gonal prism, *n* odd (Schlegel Diagram)
- My subdivision – main part

Mail route does not exist

- Single loop
- 1 x *n* array, n odd
- Tetrahedron (Schlegel diagram)
- Cube (Schlegel diagram)
- Dodecahedron (pentagonal) (Schlegel diagram)
- *n*-gonal prism, *n* even (Schlegel diagram)

A general proof?

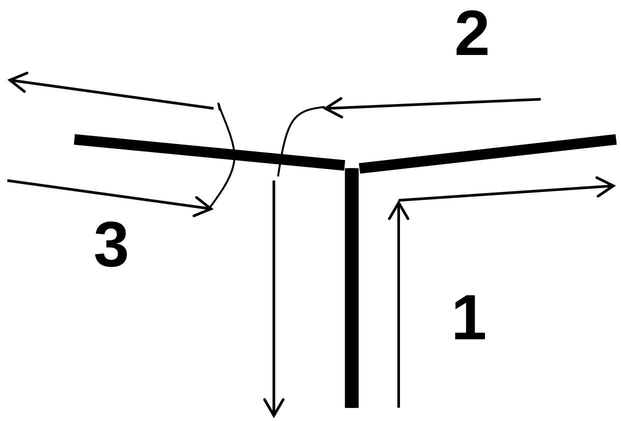
- No loops (most can't be mail routed)
- Acyclic branches (follow the right curb)
  - Cul-de-sac at beginning removes U-turn
- Degree 2 vertices
  - Can't have 2 U-turns
  - Remove vertex, and splice mail route together
- Assume all vertices are degree 3 or higher

Three Cases

1. All vertices are degree 3 (a 3-graph)
2. All vertices are degree 4 or higher
3. Some vertices of degree 3 and some higher degree (mixed graph)

Case 1

example has U-turn

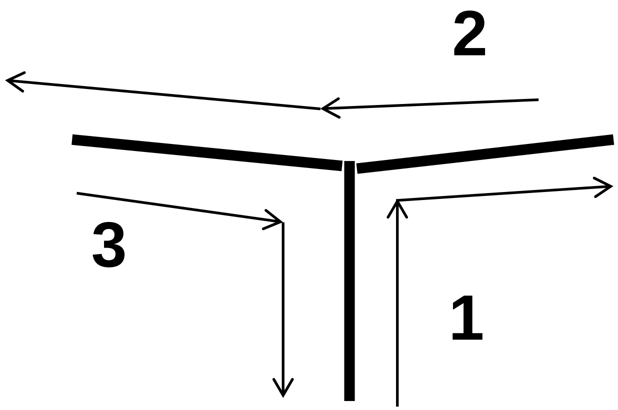


$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}$$

Vertex Permutation

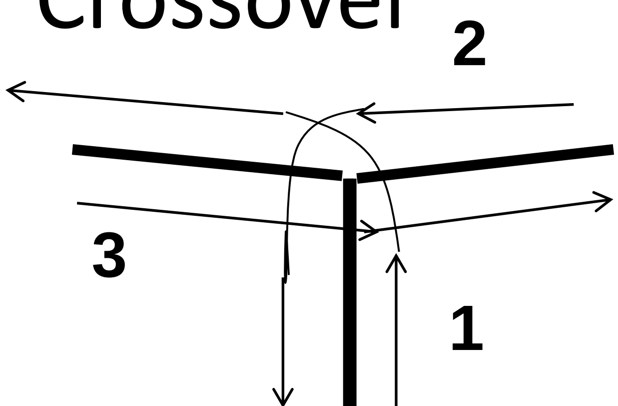
Case 1

Standard



$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$$

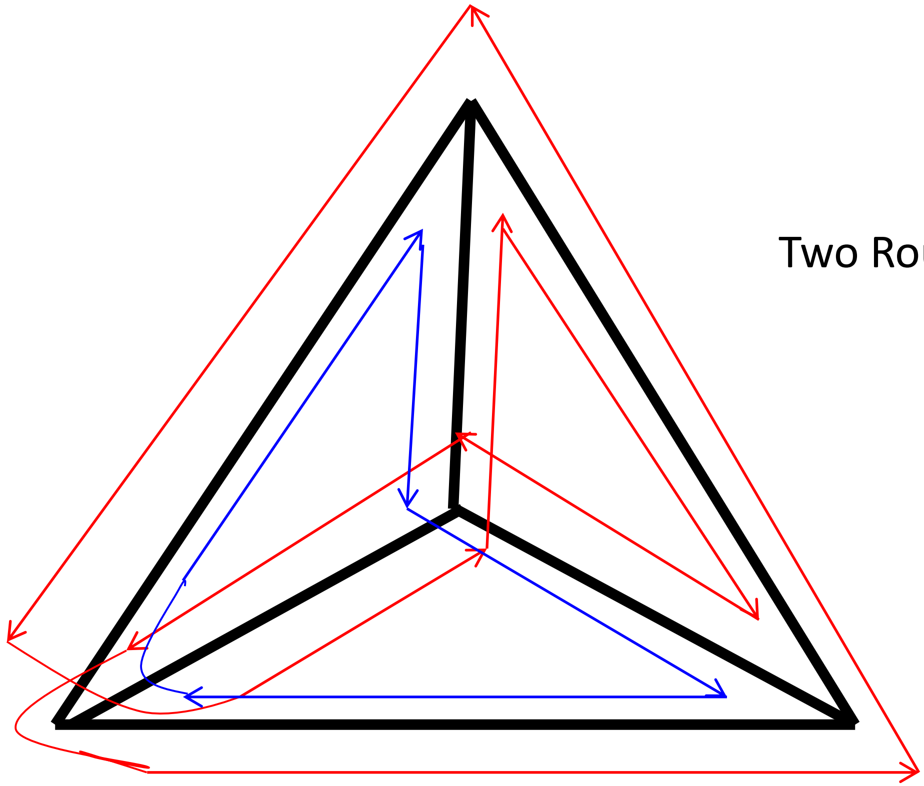
Crossover



$$\begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}$$

More than one subroute

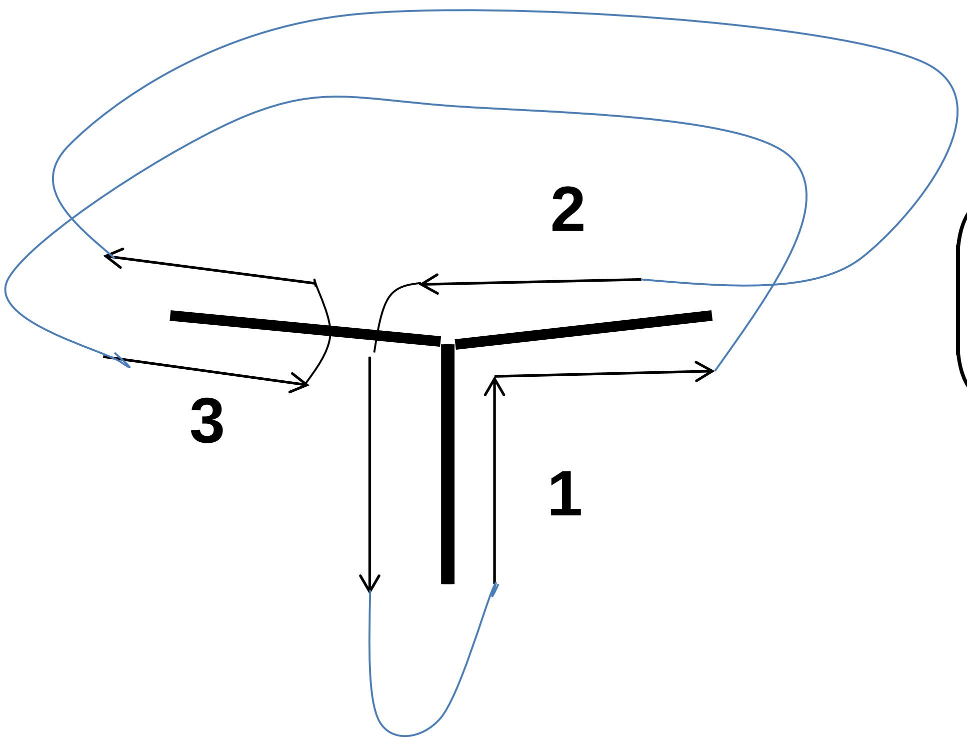
tetrahedron



Two Routes: blue and red

Not acceptable – we have only one truck.

Excursions



VE=P

$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} * \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}$$

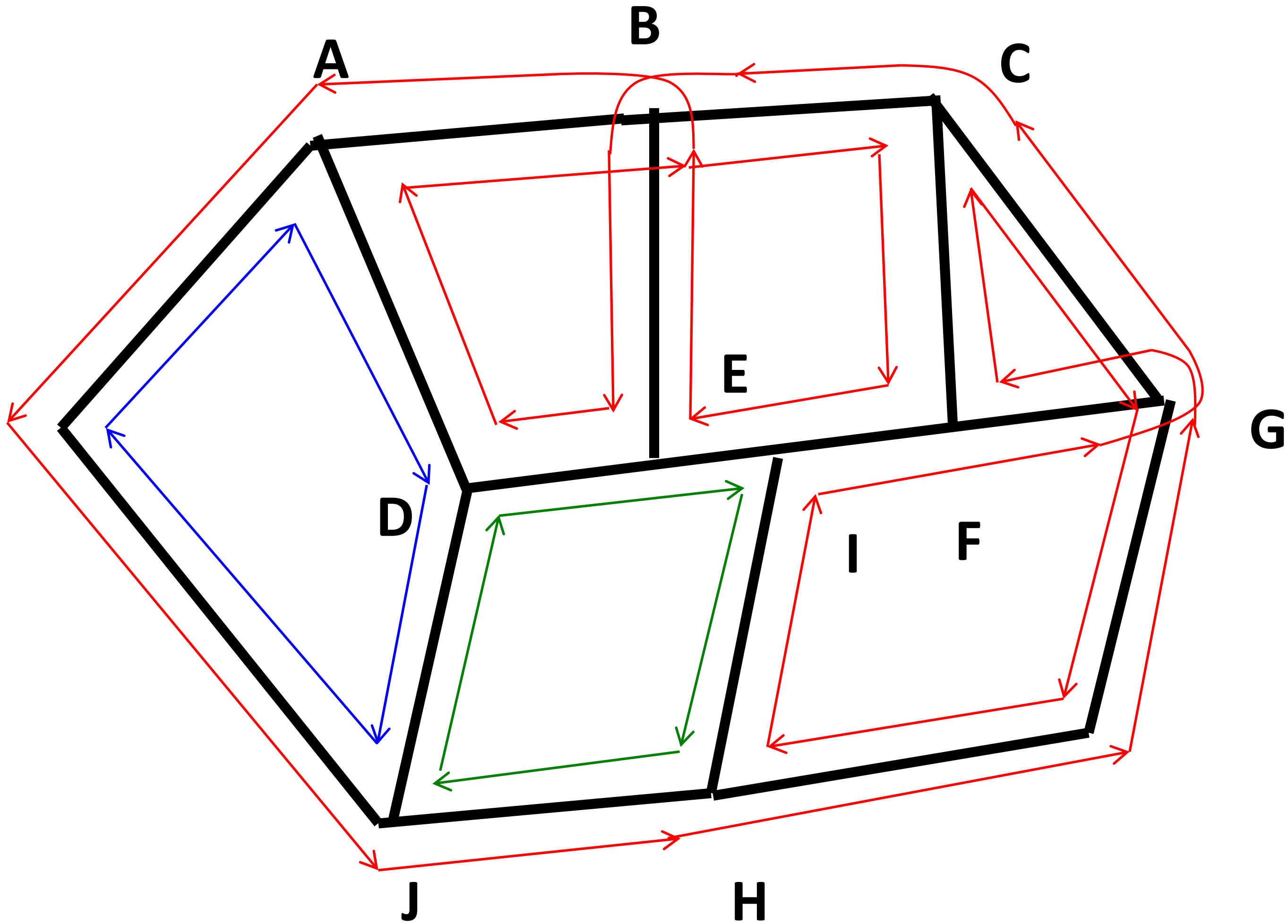
$$= \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}$$

Theorem

- *A connected planar 3-graph with an odd number of regions is not mailroutable.*
- Proof: will be illustrated with example

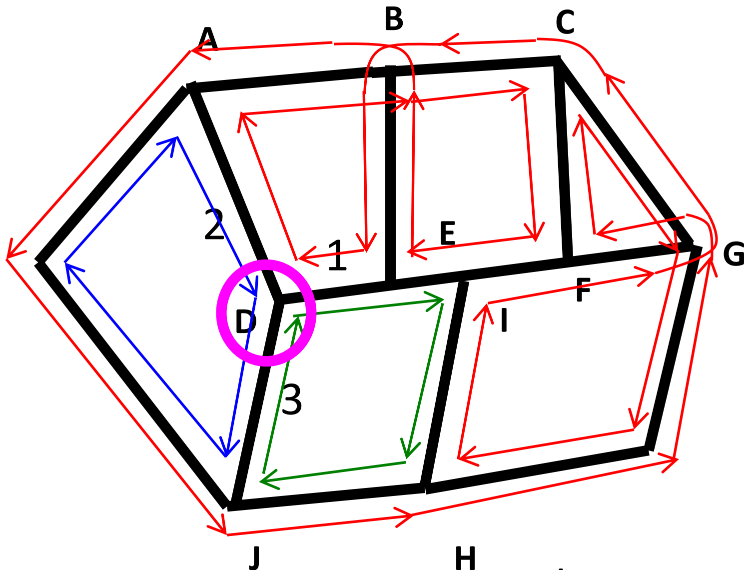
Example

all vertices standard except B and G



Example
all vertices standard except B and G

Examine Vertex D



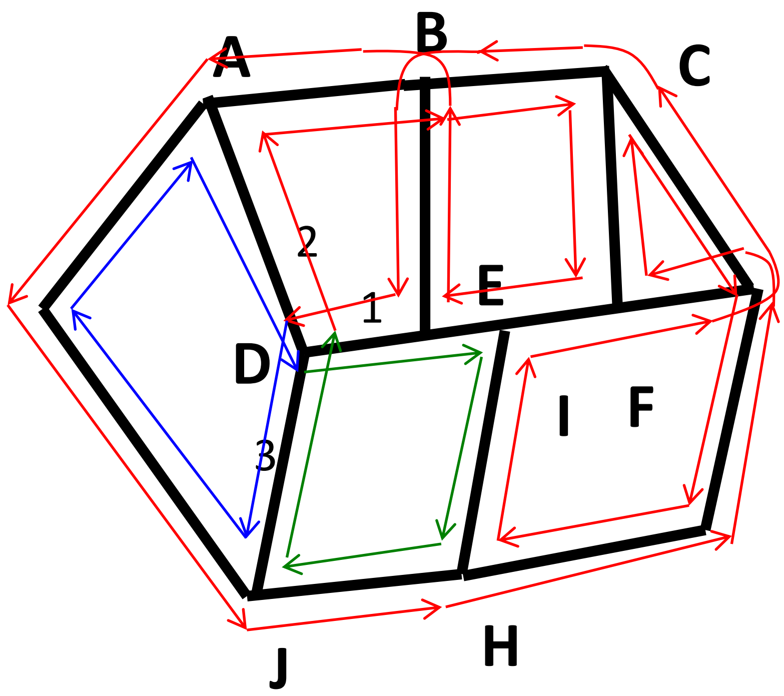
Three Routes

$$V = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$$

$$E = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}$$

$$P = VE = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}$$

Example
all vertices standard except B and G



One Route
P is a 3-cycle

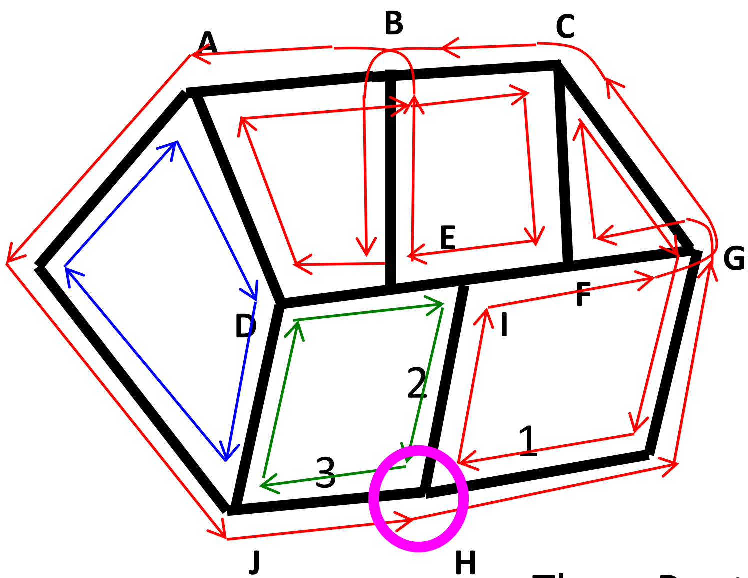
$$V = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}$$

$$E = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}$$

$$P = VE = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$$

Example
all vertices standard except B and G

Examine Vertex H



Three Routes

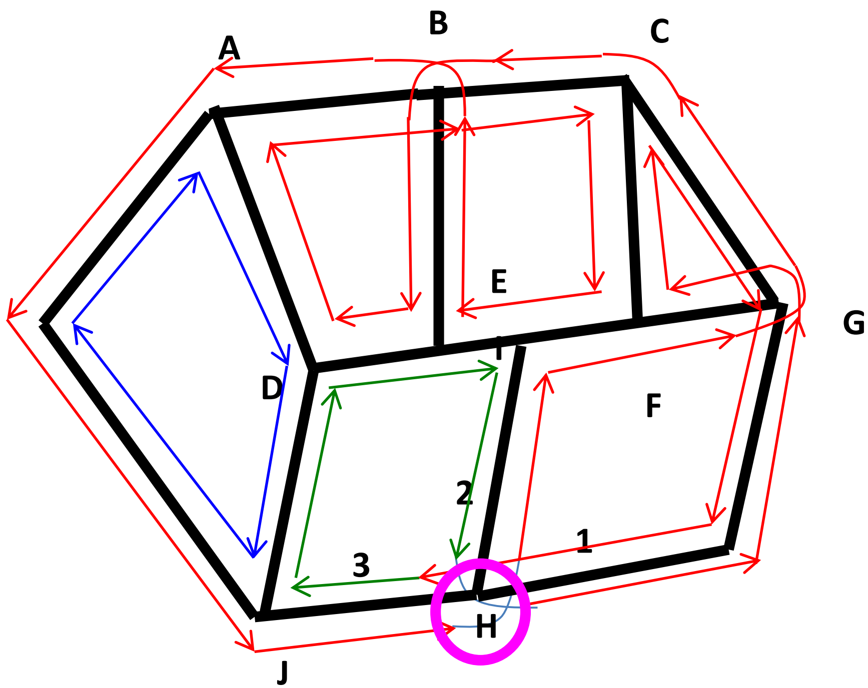
$$V = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$$

$$E = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}$$

$$P = VE = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}$$

Example

tweak Vertex H (which isn't full)



$$V = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}$$

$$E = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}$$

Two Routes

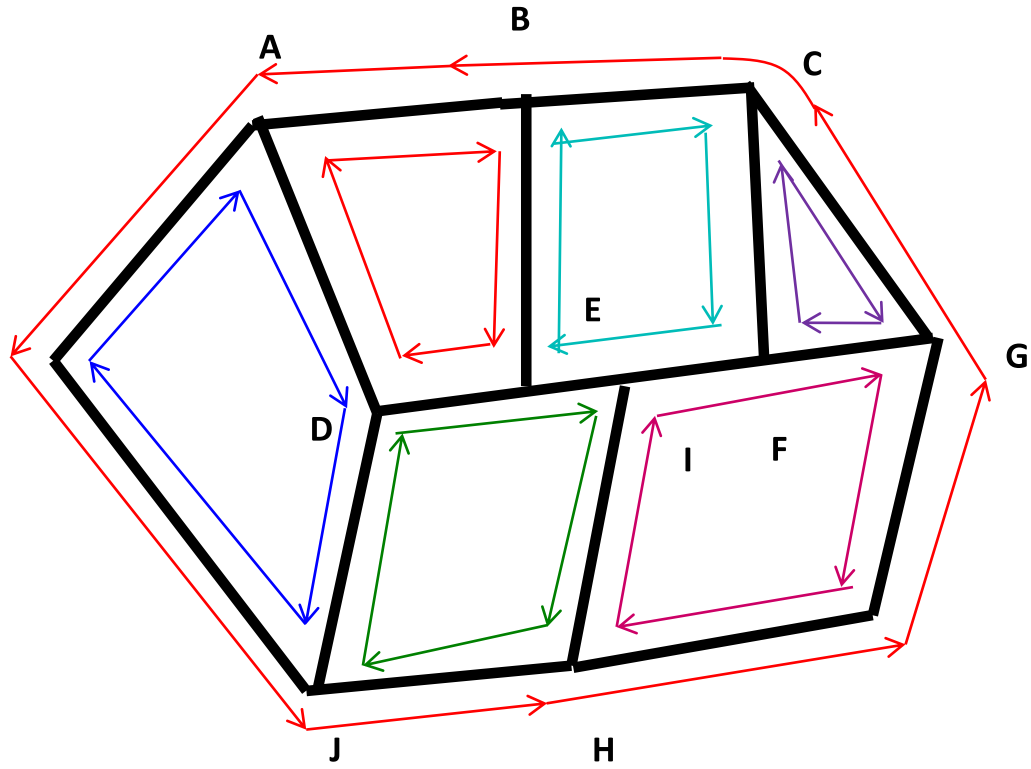
$$P = VE = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}$$

Parity of number of routes is invariant

- P = VE
  - Therefore, $E = V^{-1}P$
- For there to be one route
  - V^{-1}, and hence V, must have no fixed points
  - P must be an *n*-cycle
- If $n=3$, then no-fixed-point and *n*-cycle permutations both are in A_3.

Initial Attempt

all intersections standard



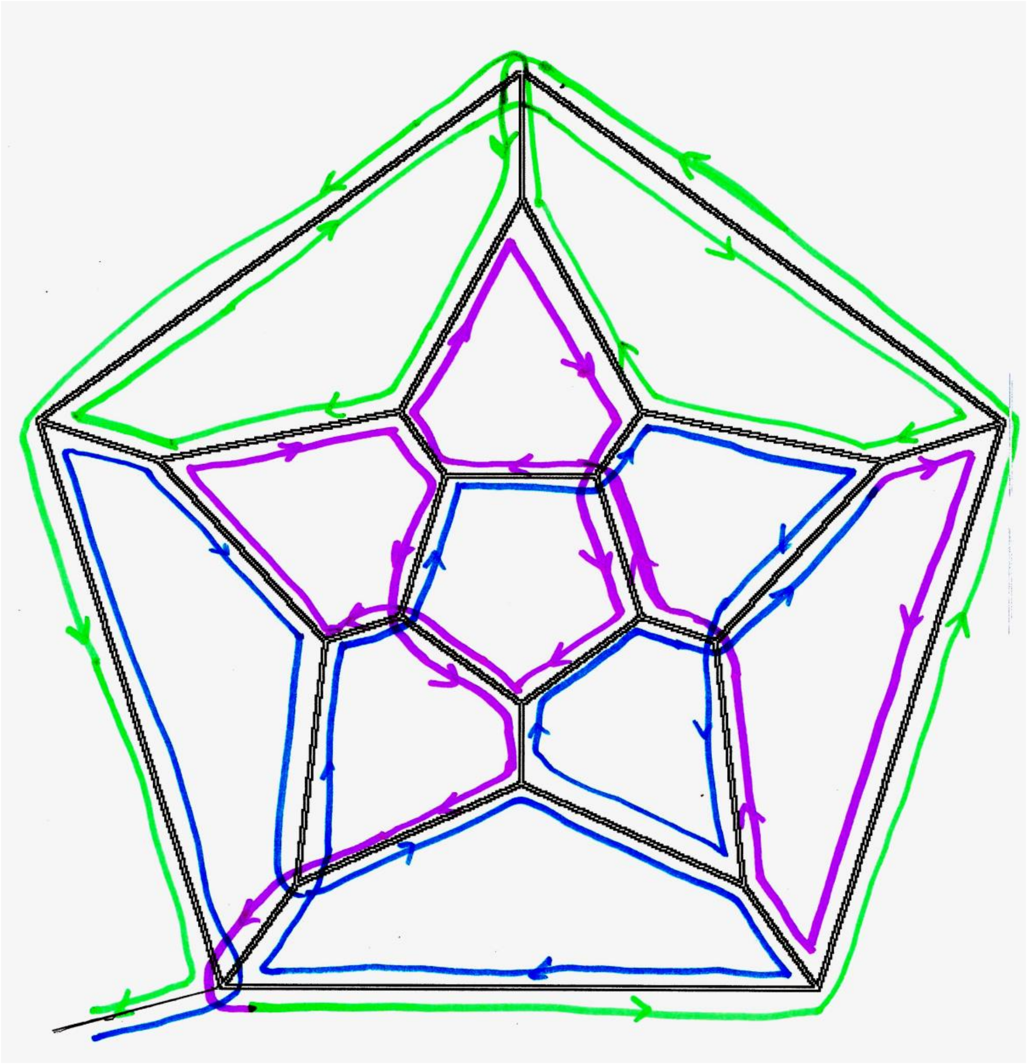
Let r be the number of regions. Then r+1 is the number of cycles in this mailroute attempt. If r is odd, r+1 is even, and number of cycles is always even, so never 1.

Conclusion

- If there are an odd number of regions, and every vertex has order 3, there is no mail route.
- End of proof.
- But not all is lost!
- There is a fudge

Dodecahedron

Mail route with access road

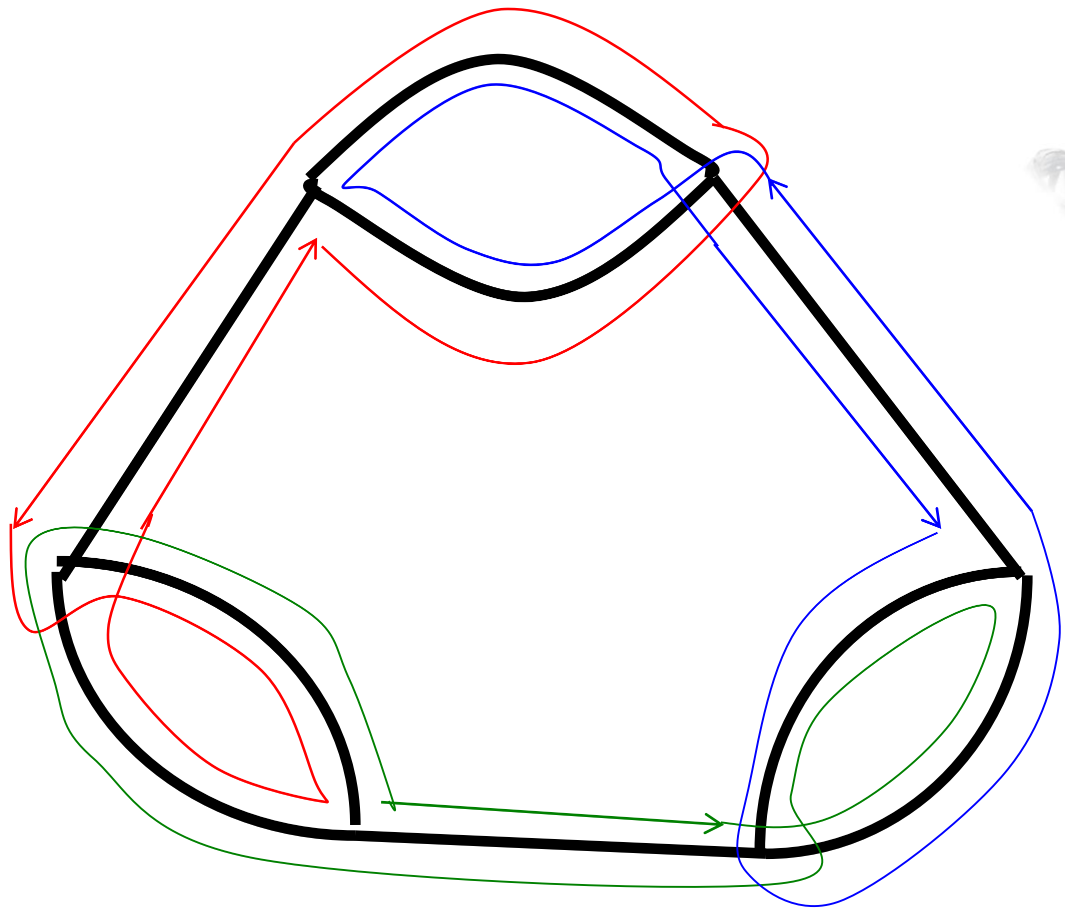


How about even number of regions?

- Most often there is a solution
- Draw an even-regioned mail route example, and almost certainly there is a route for it without U-turns
- BUT NOT ALWAYS!

Counterexample

4 regions, but cannot be mailrouted: contains goggles



Excursions

- *Excursion Domain* of vertex v is the set of oriented edges that are on subroutes that are incident to v.
- A vertex is *full* if its excursion domain is the entire graph.

Theorem

- Suppose G is a connected oriented graph with an even number of regions and all vertices of degree 3, and suppose there is an assignment of “standard” and “crossover” to the vertices of the graph such that there are exactly three subroutes, and there exists a full vertex v with all three subroutes going through v. Then G is mailroutable.

Case 2

All vertices degree 4 or higher

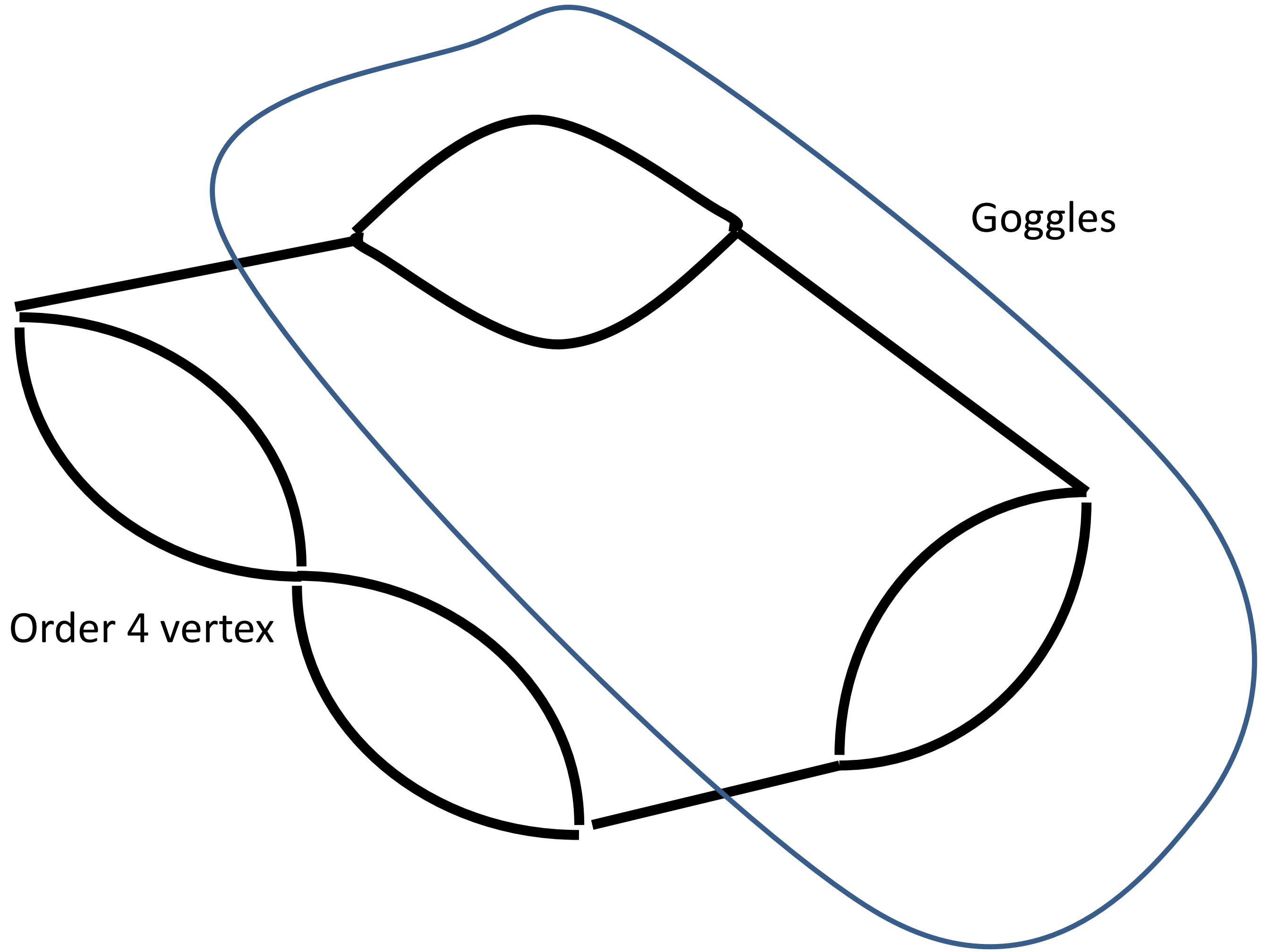
- Rare in practice; usually boundaries have 3-vertices.
- Theorem of Carraher and Hartke implies that if a connected planar graph has all vertices of degree 4 or higher, then it is mailroutable.

Case 3

Mixed connected planar graphs - Strategy

- Change a 4+-vertex so that the number of subroutes is odd
- Tweak vertices where three routes come together to make them into one route
- Continue doing this until there is only one subroute.
- This is now a route for the graph.
- Unfortunately, there is a counterexample here, too.

Mixed graph - Counterexample



Conjecture

- Given a connected planar graph G for which either
  - there are an even number of regions and all vertices are of degree 3
  - Or there is at least one vertex of degree 4 or higher
- And G contains no goggles
- Then G is mailroutable.

Future opportunities for research

- Classify all mailroutable 3-vertex graphs with an even number of edges
- Classify all mailroutable mixed graphs
- Prove the no-fixed-points-n-cycle conjecture
- Minimize left turns
  - FACE HITTING SET – could be NP-complete
  - Attach cost to U and left turns and minimize cost
- Introduce one-way streets
  - No longer have Eulerian circuits
  - CHINESE POSTMAN

Conclusion

- Planar graphs with an odd number of regions and all vertices of degree 3 are not mailroutable.
- Planar graphs with all vertices of degree 4 or higher are mailroutable.
- Lots of unanswered questions provide fertile grounds for research.
- May you get your mail on time.

END OF PRESENTATION