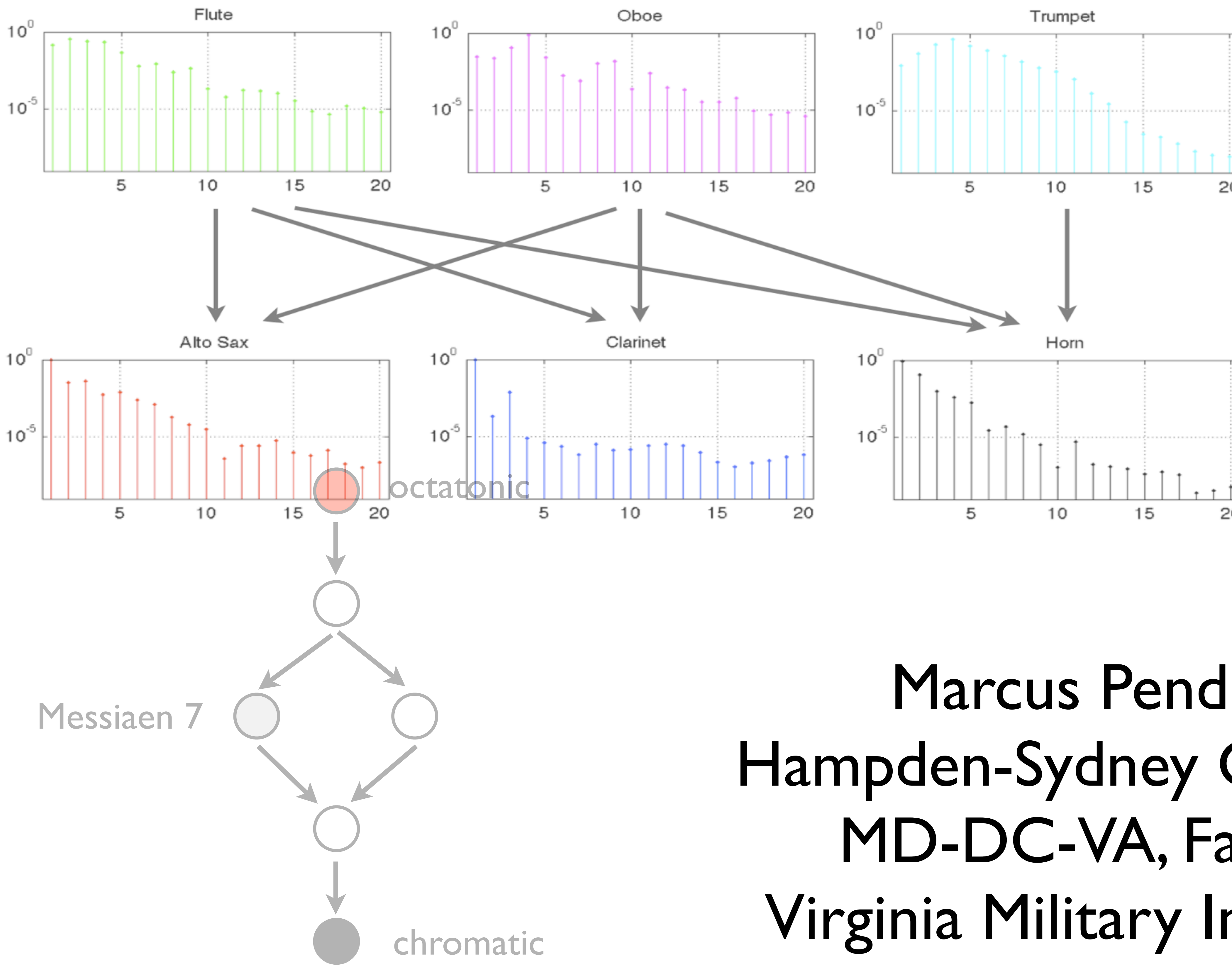
# Two Musical Orderings

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Hampden-Sydney College
MD-DC-VA, Fall 2012
Virginia Military Institute

# Partial Orders in Music Theory

- Submajorization and Voice Leading
  - Tymoczko (2004, 2008), Callendar, Quinn and Tymoczko (2008), Hall and Tymoczko (2012)
- Set Inclusion and Pitch Class Sets
  - Straus (2005)
- New Ideas
  - Extemal elements and harmony
  - Stochastic dominance and timbre

# Some Connections

- Submajorization and Voice Leading (Tymoczko)
- Orbifolds and Musical Geometry (Tymoczko)
- The Geometry and Topology of Three-Manifolds (Thurston, 1980)
- Geometrization Conjecture (Thurston, 1982)
- Perelman's proof of the Poincare Conjecture (Perelman, 2003)
- Perelman refuses Fields Medal (2006) and Clay Millenium Prize (Perelman, 2010, $\$10^6$)

# Orderings

- A *partial order* on a set $S$ is a relation $\leq$ that is
  - Reflexive: $a \leq a$ for all $a \in S$
  - Transitive: $a \leq b$ and $b \leq c$ implies $a \leq c$ for all $a, b, c \in S$
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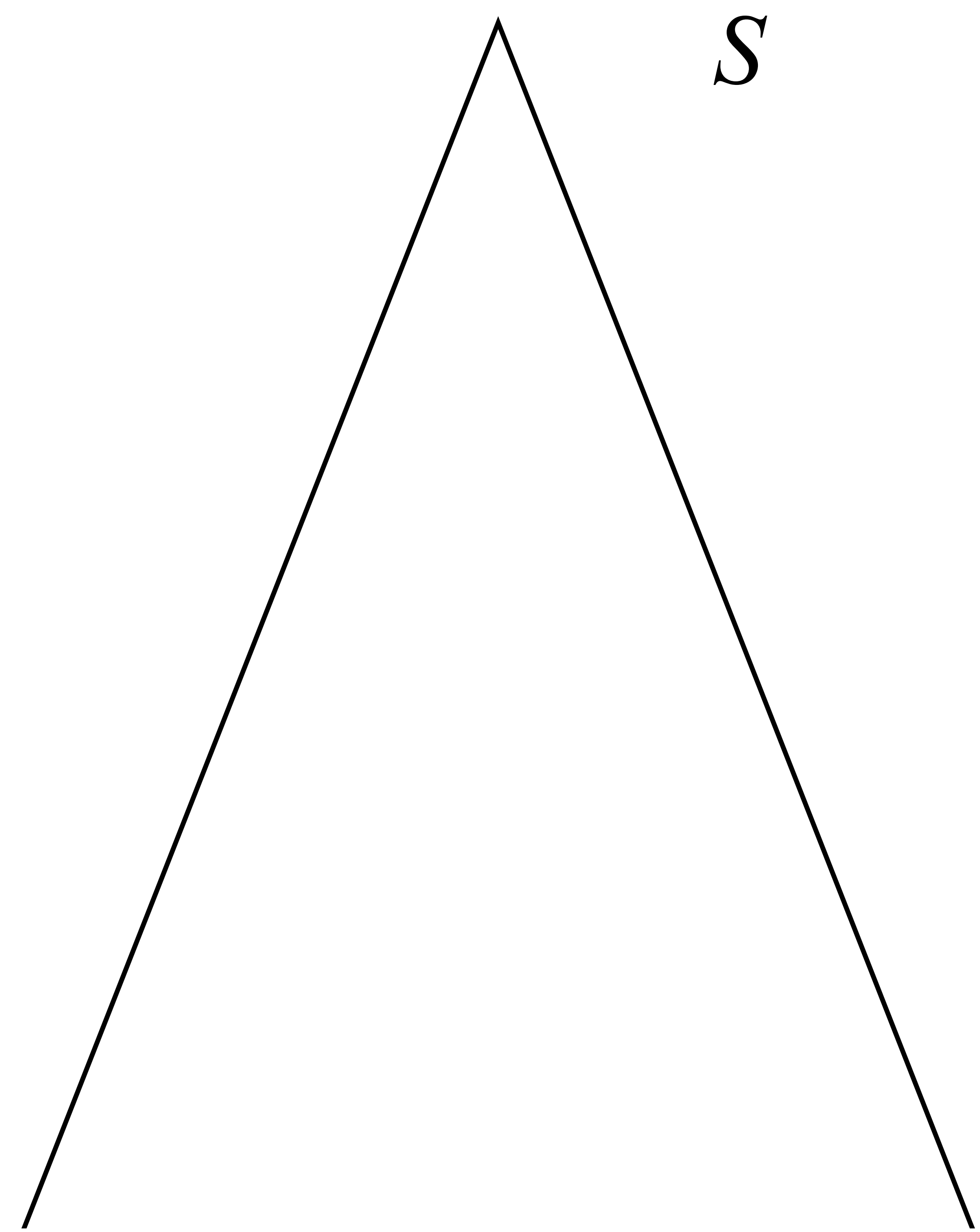

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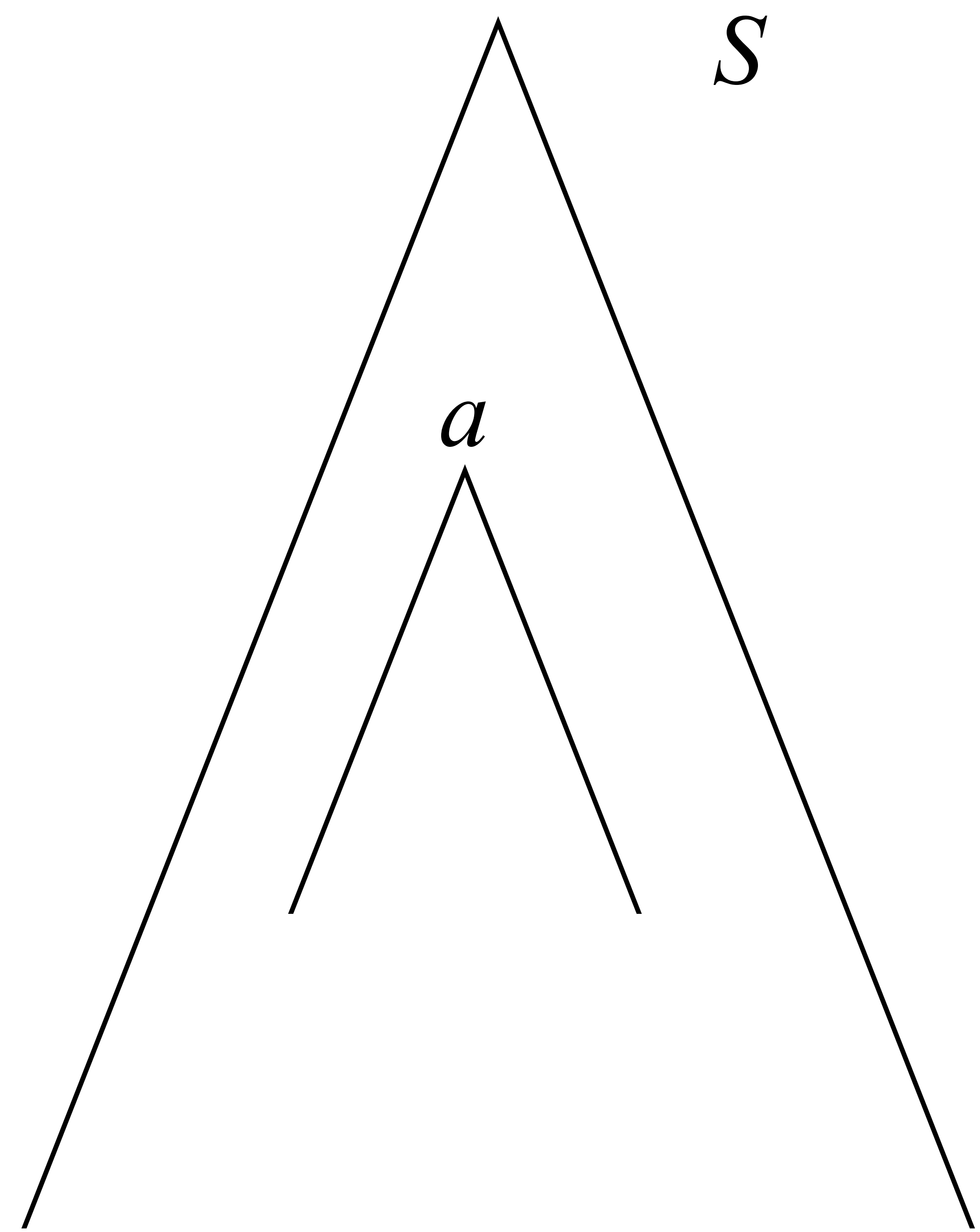

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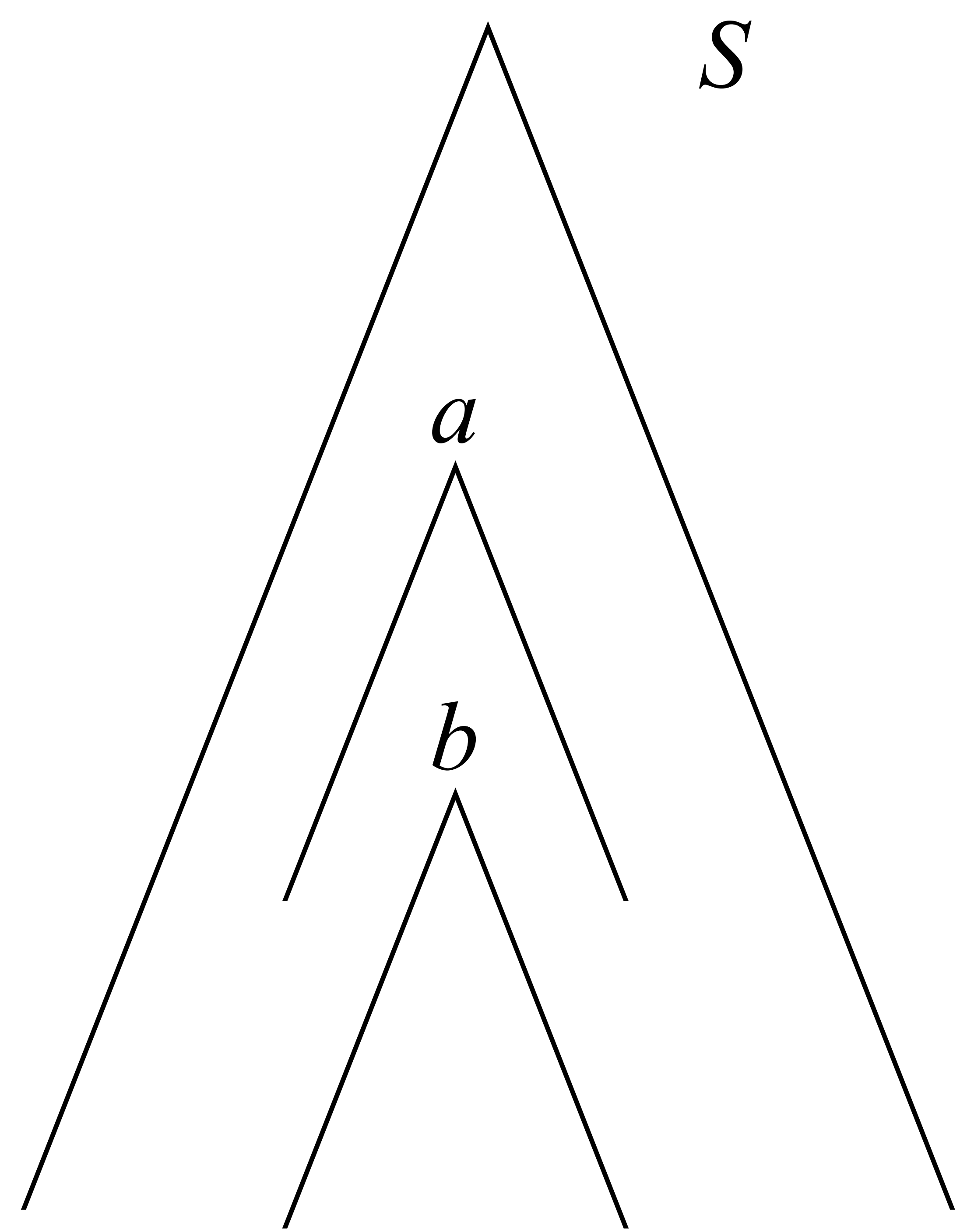

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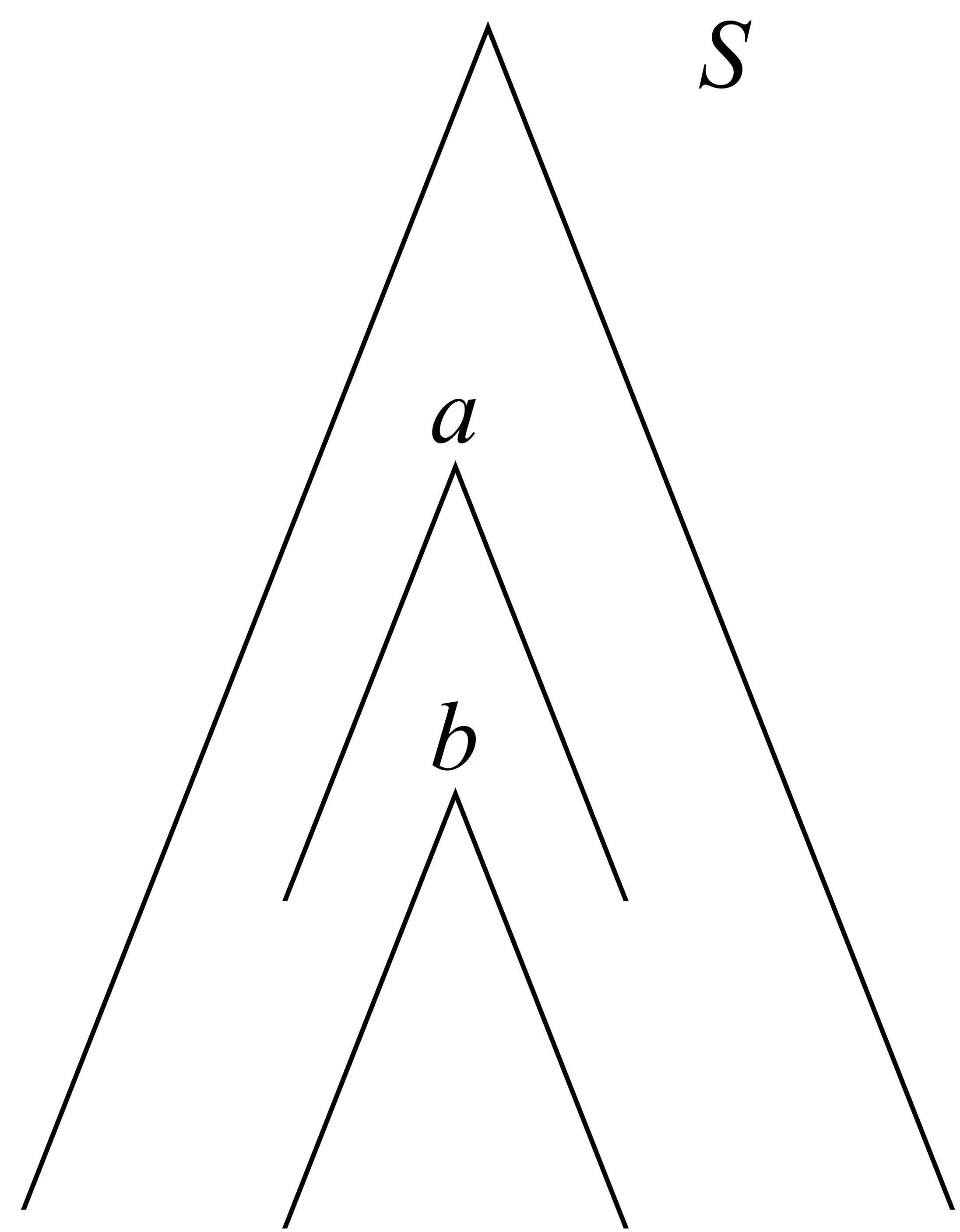


$$a \leq b$$

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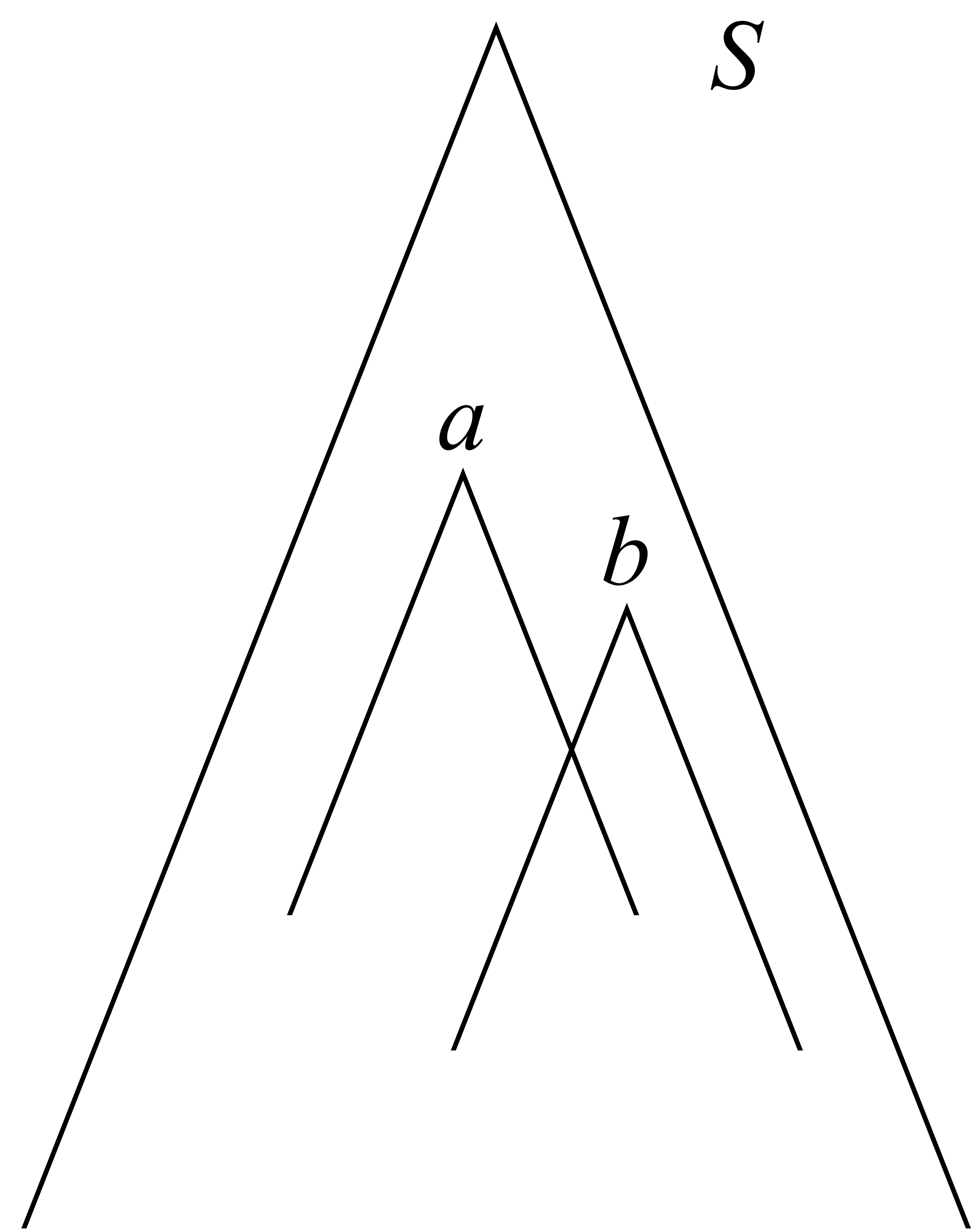
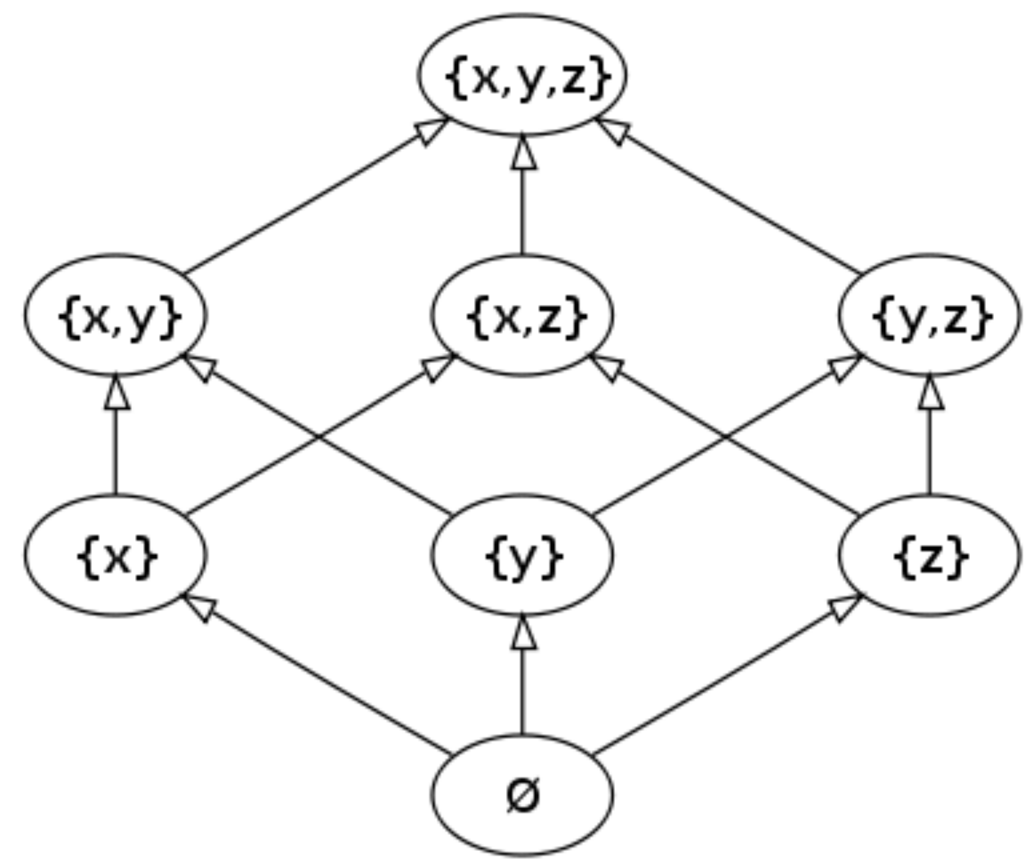



$a, b$ incomparable

# Set Inclusion Ordering

- Partial order induced by set inclusion.
- Music theory: scales and harmony
  - C pentatonic: {C, D, E, G, A}
  - C diatonic: {C, D, E, F, G, A, B}
  - C pentatonic $\subseteq$ C diatonic




- Theme: partial order models some notion of size or precedence among musical objects

# Partial Order modulo Group

- Often we want to identify certain musical objects as being essentially "the same":
  - All diatonic scales are "the same": C major ≡ G major ≡ ...
    - "transpositional equivalence"
  - All notes separated by whole octaves are "the same": middle A ≡ high A ≡ ...
    - "octave equivalence"

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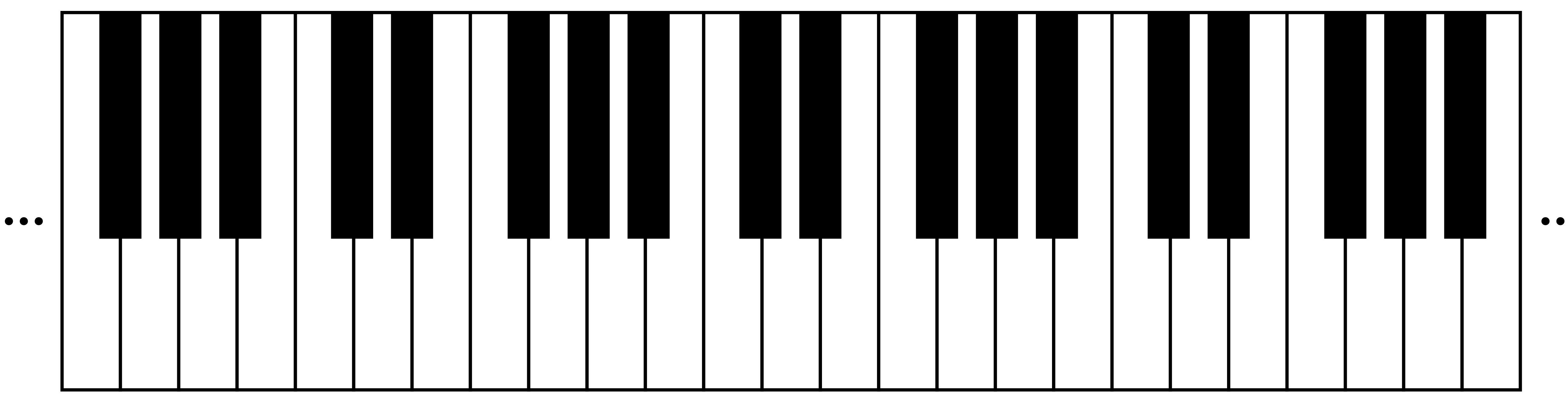

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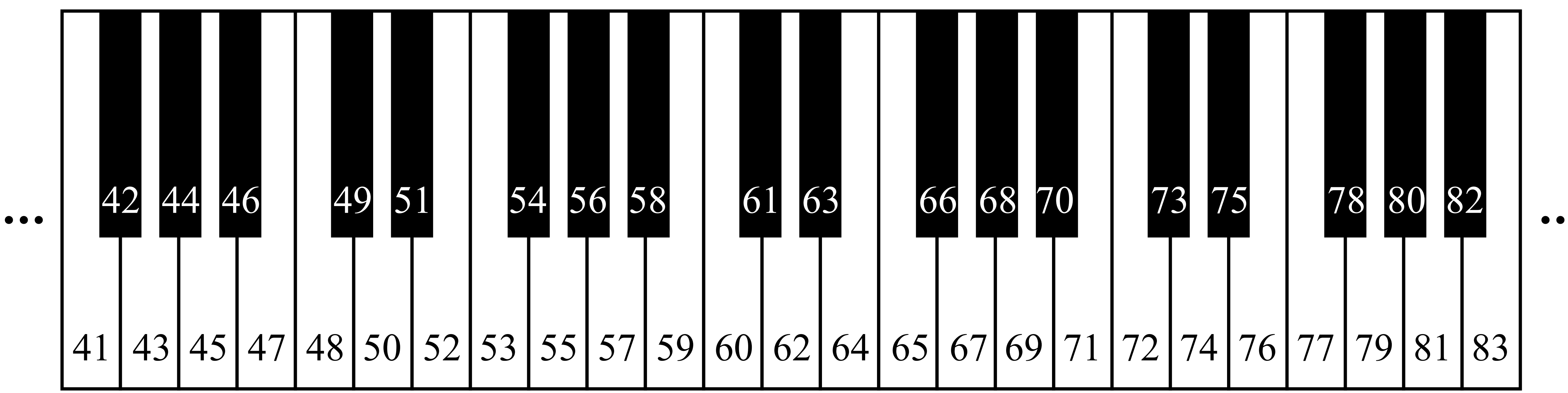

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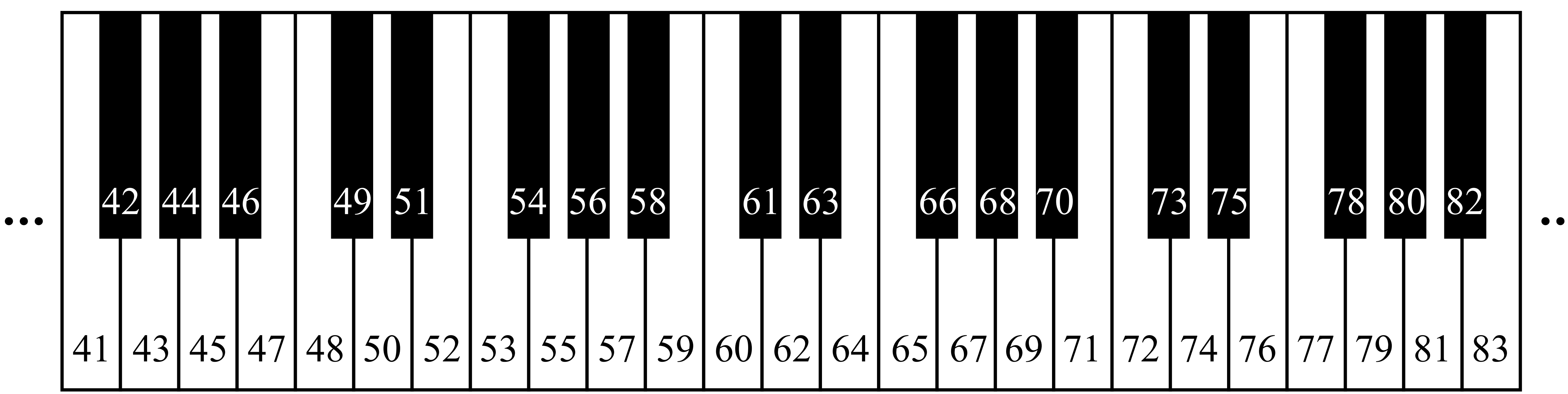
- $S = \mathbb{Z}$ = "infinite keyboard"
- $G = < z + 12 >$ = " octave equivalence"



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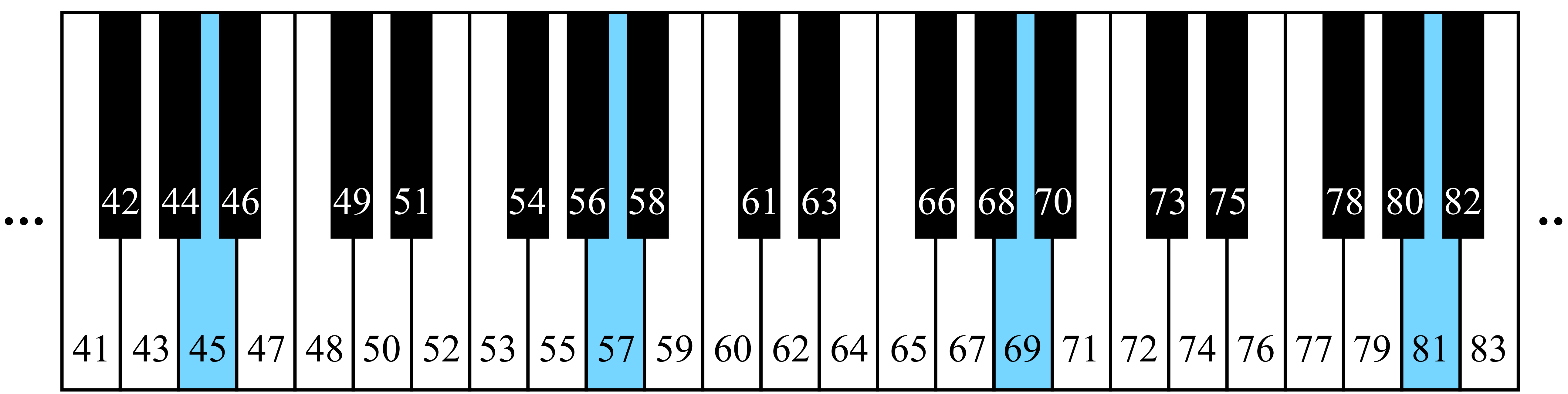

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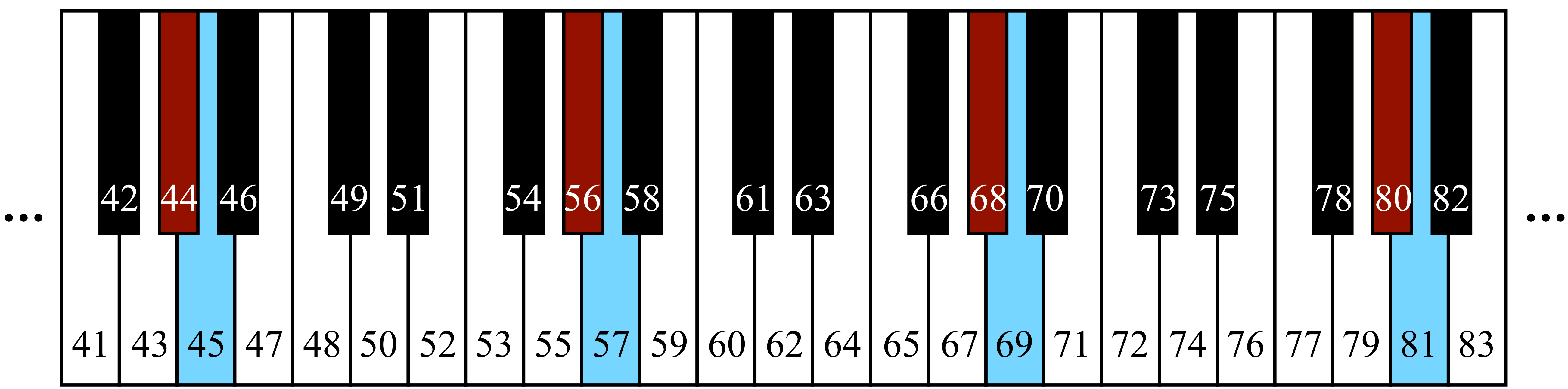

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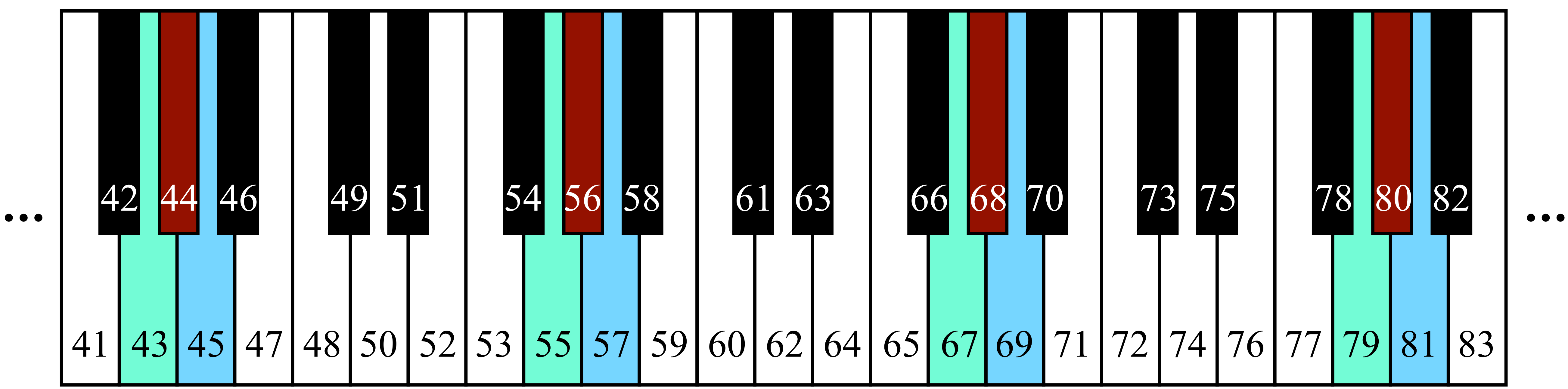

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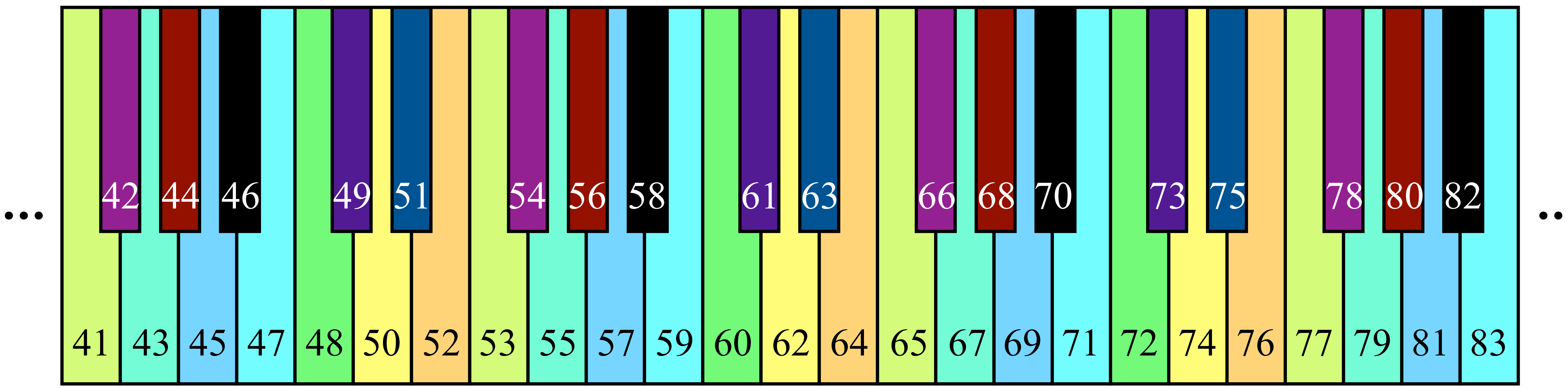

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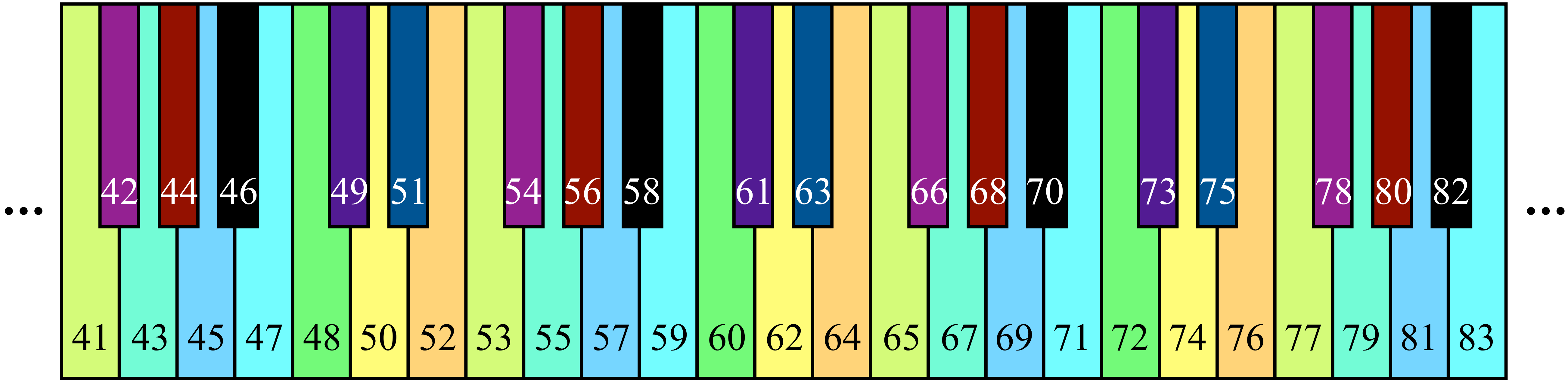

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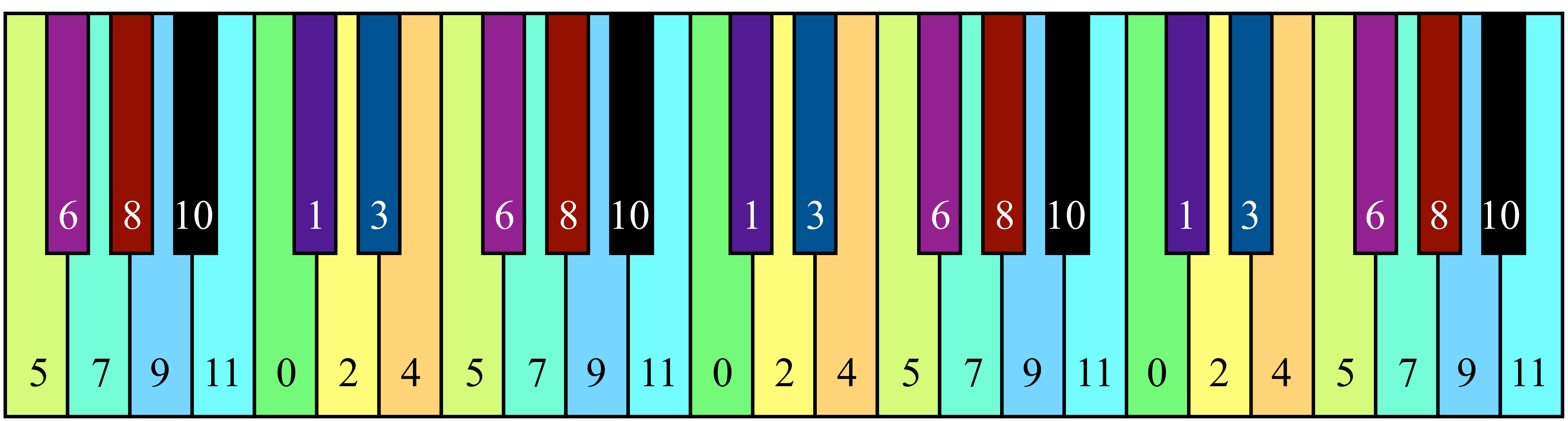


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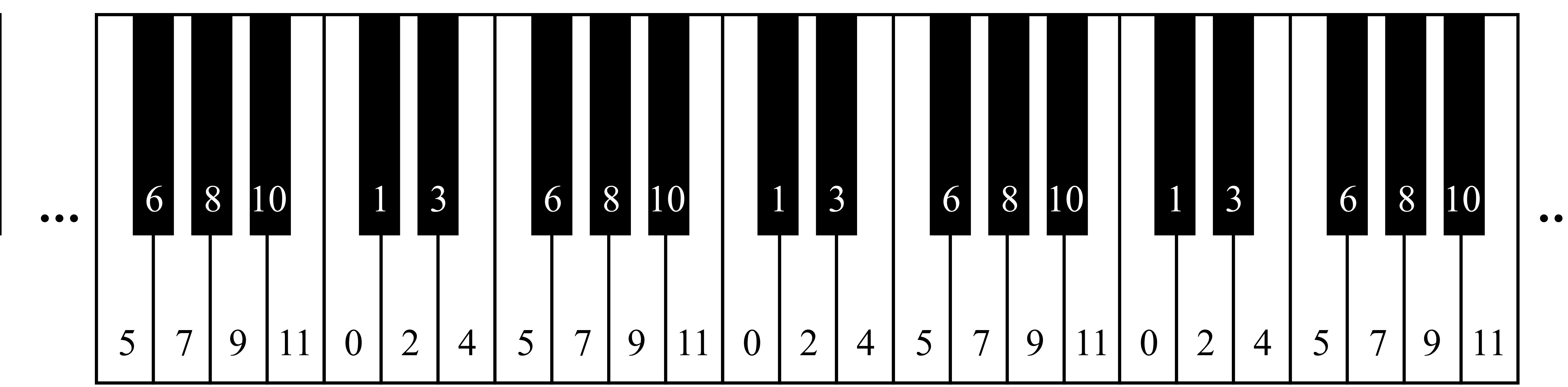

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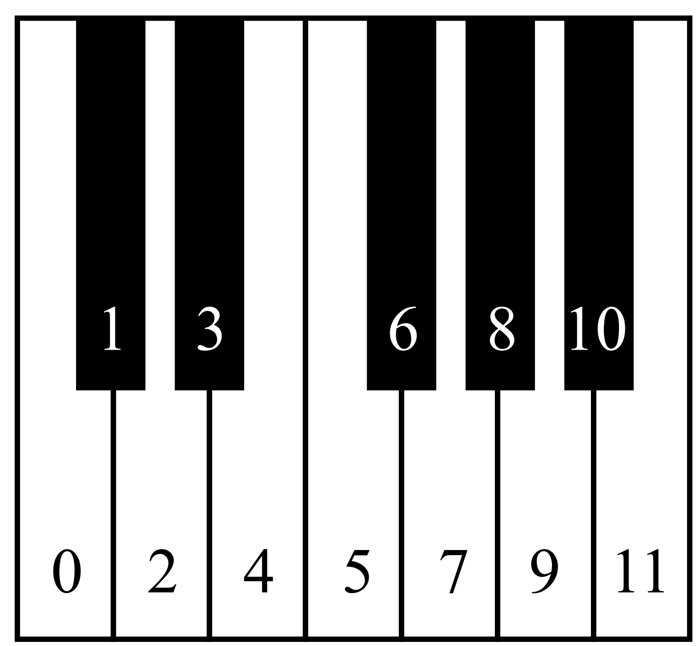

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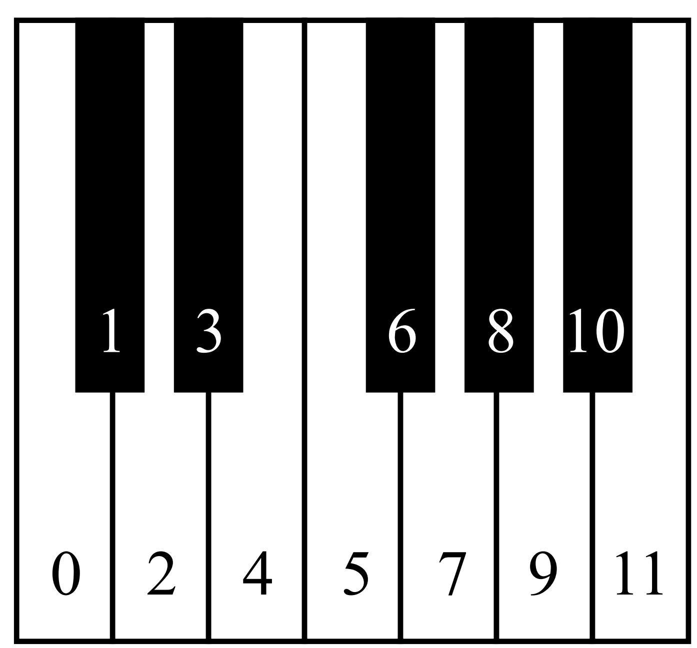

# Partial Order modulo Group

- We may also want to model some notion of musical "motion" or "transformation":
  - Transpose all notes up one octave.
  - Move from the tonic to the dominant.

# Partial Order modulo Group

- A group $G$ acting on $S$ can serve both purposes:
  - Equivalence: $a, b \in S$ are "the same" if $b = Ta$ for some $T \in G$.
  - Motion: can "move" from $a$ to $b$ if $b = Ta$ for some $T \in G$

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How do the group and the partial order interact?

# Partial Order modulo Group

- Equivalence classes
  - $A = [a] =$ set of all $b \in S$ that are essentially "the same" as $a$.

    $= \{b \in S : b = Ta$ for some $T \in G\}$
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  - $S / G =$ set of all distinct equivalence classes.
- Induced relation on $S / G$
  - $A \leq B$ if and only for all $x \in A$ there exists $y \in B$ such that $x \leq y$

# Partial Order modulo Group

**Theorem 1**

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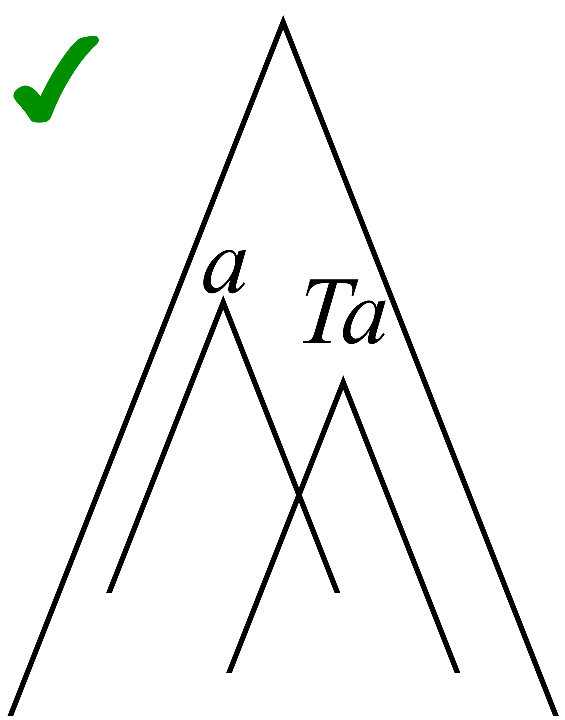

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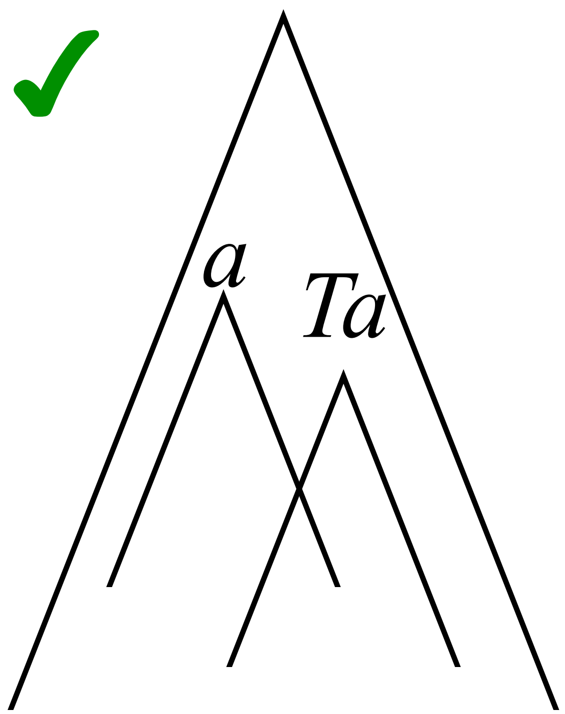


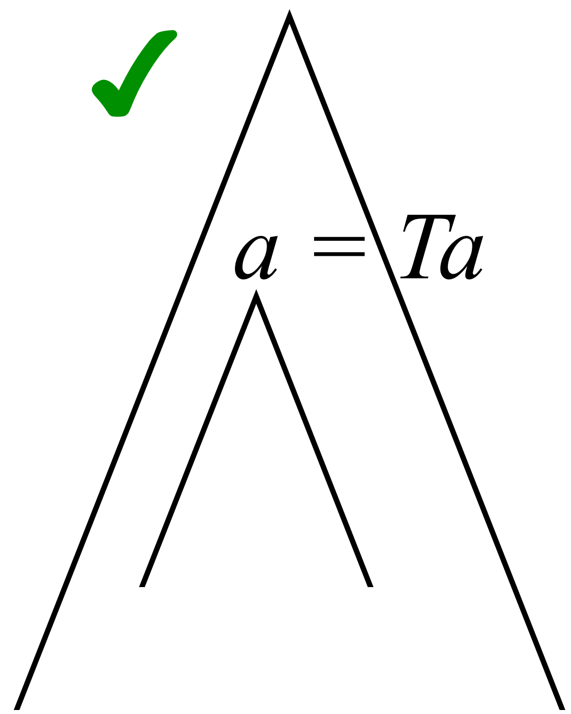

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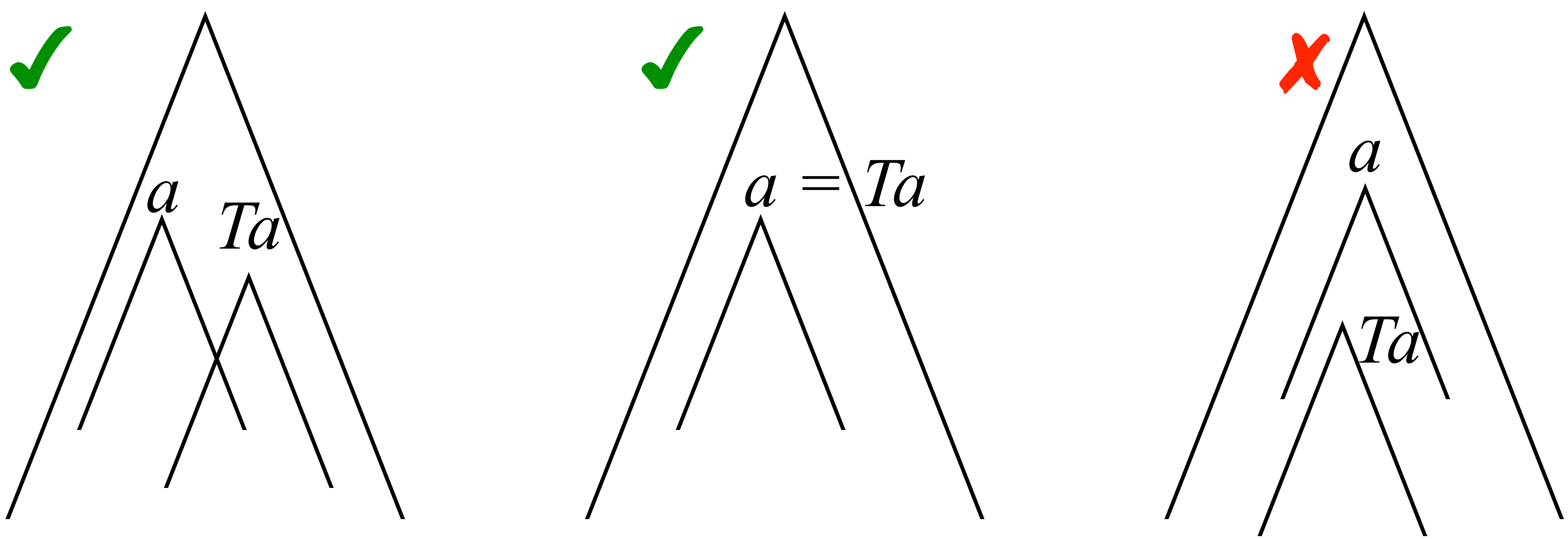

# A Partial Order on Scales

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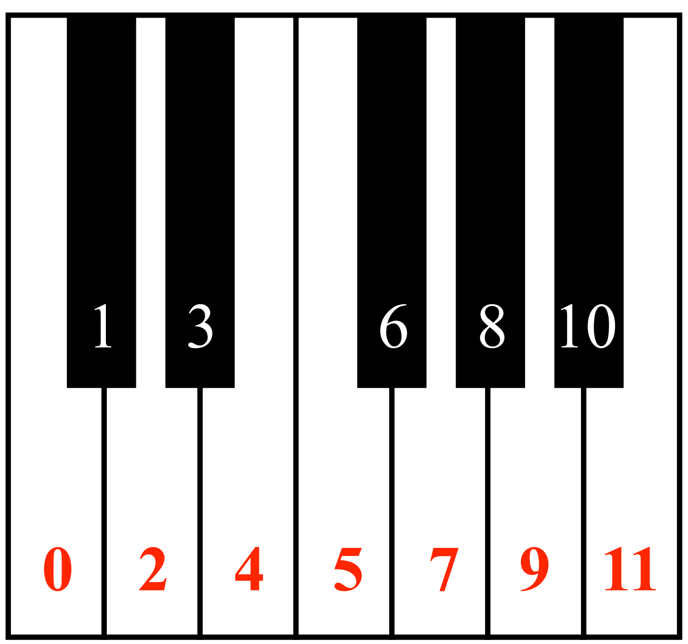
diatonic




$(2, 2, 1, 2, 2, 2, 1)$

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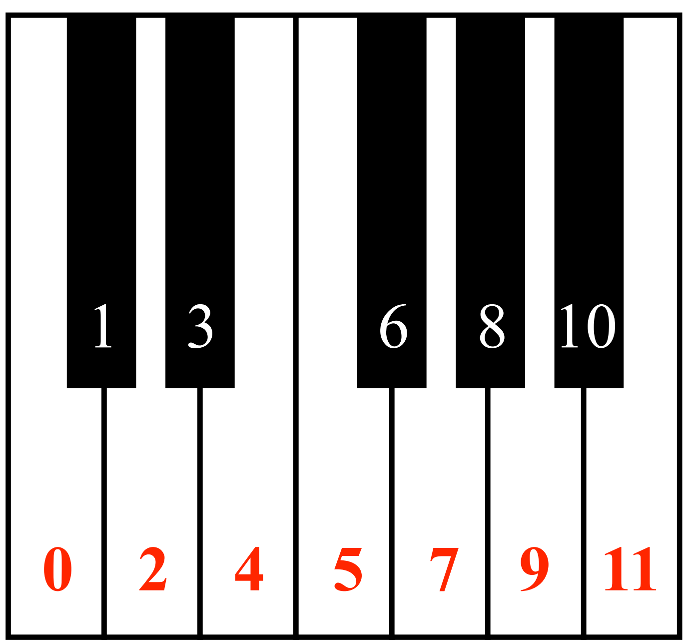
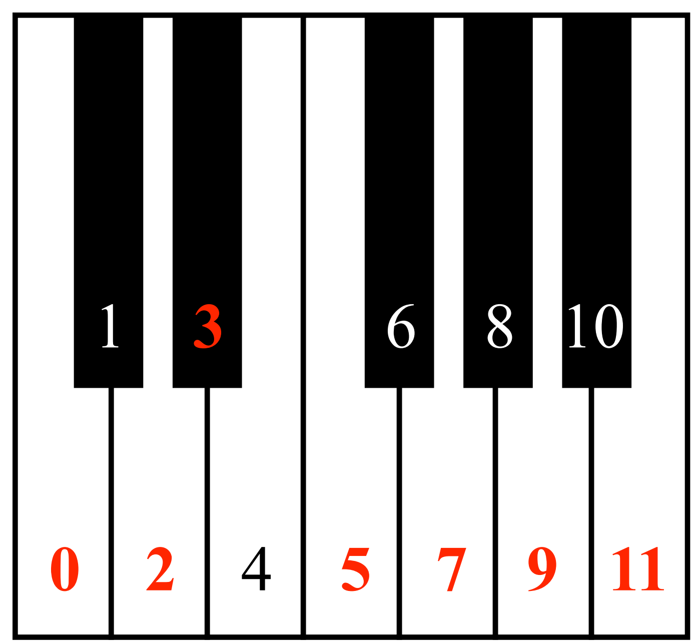



$(2, 2, 1, 2, 2, 2, 1)$

melodic minor

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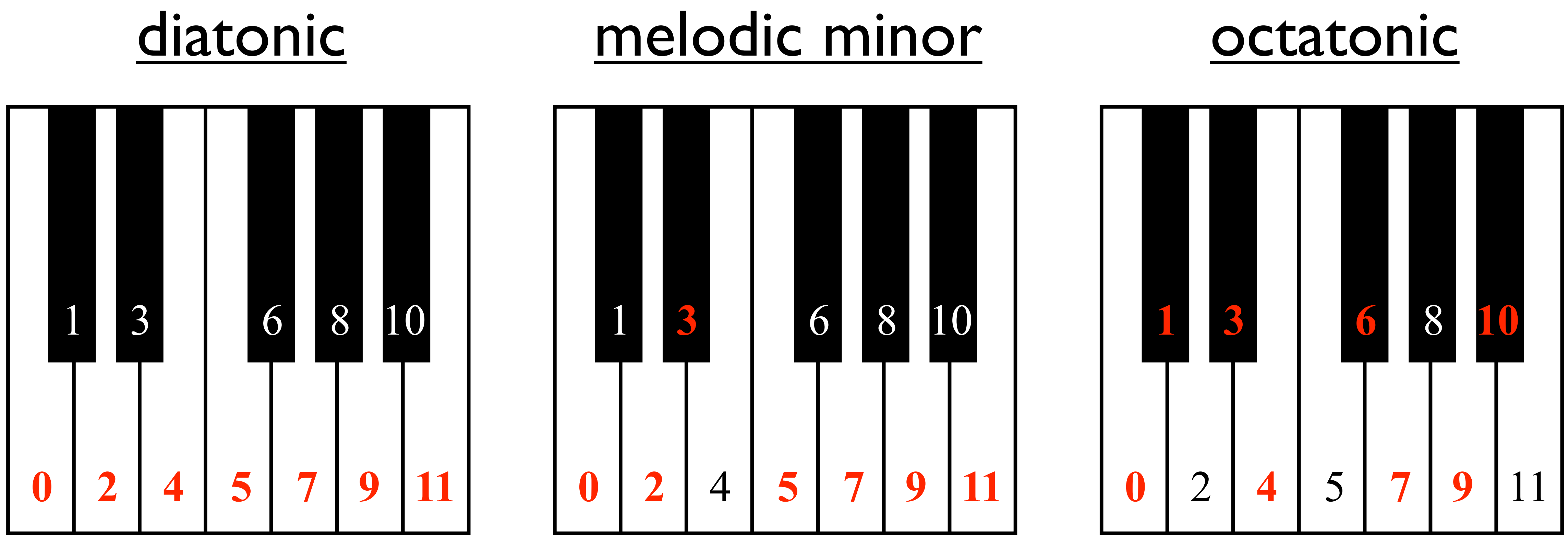


(2, 2, 1, 2, 2, 2, 1) (2, 1, 2, 2, 2, 2, 1) (1, 2, 1, 2, 1, 2, 1, 2)

# A Partial Order on Scales

- Equivalences
  - Octave equivalence
  - Transpositions (translations) of a scale are all “the same”
    - C major ≡ G major ≡ D major ≡ ...
  - Modes (rotations) of a scale are all “the same”
    - (2,2,1,2,2,2,1) ≡ (2,1,2,2,2,1,2) ≡ (1,2,2,2,1,2,2) ≡ ...

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✓ $G$ acts transversely on $S$

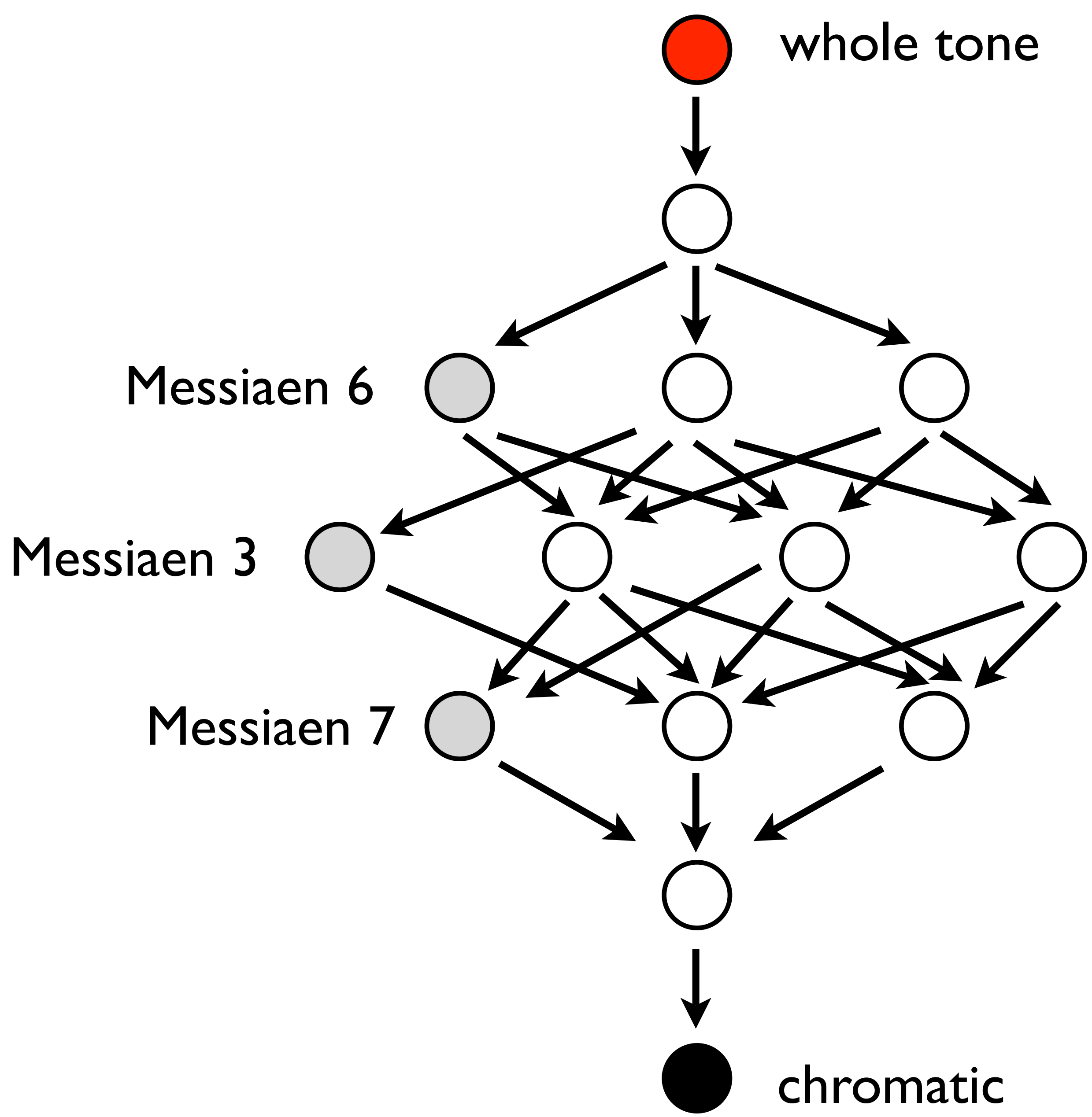
✓ $S / G$ is a partial order

# A Partial Order on Scales

- $S\,/\,G$ contains $31$ distinct scales.
- $S\,/\,G$ has four *minimal elements*:
    - whole tone scale
    - diatonic scale
    - melodic minor scale
    - octatonic scale

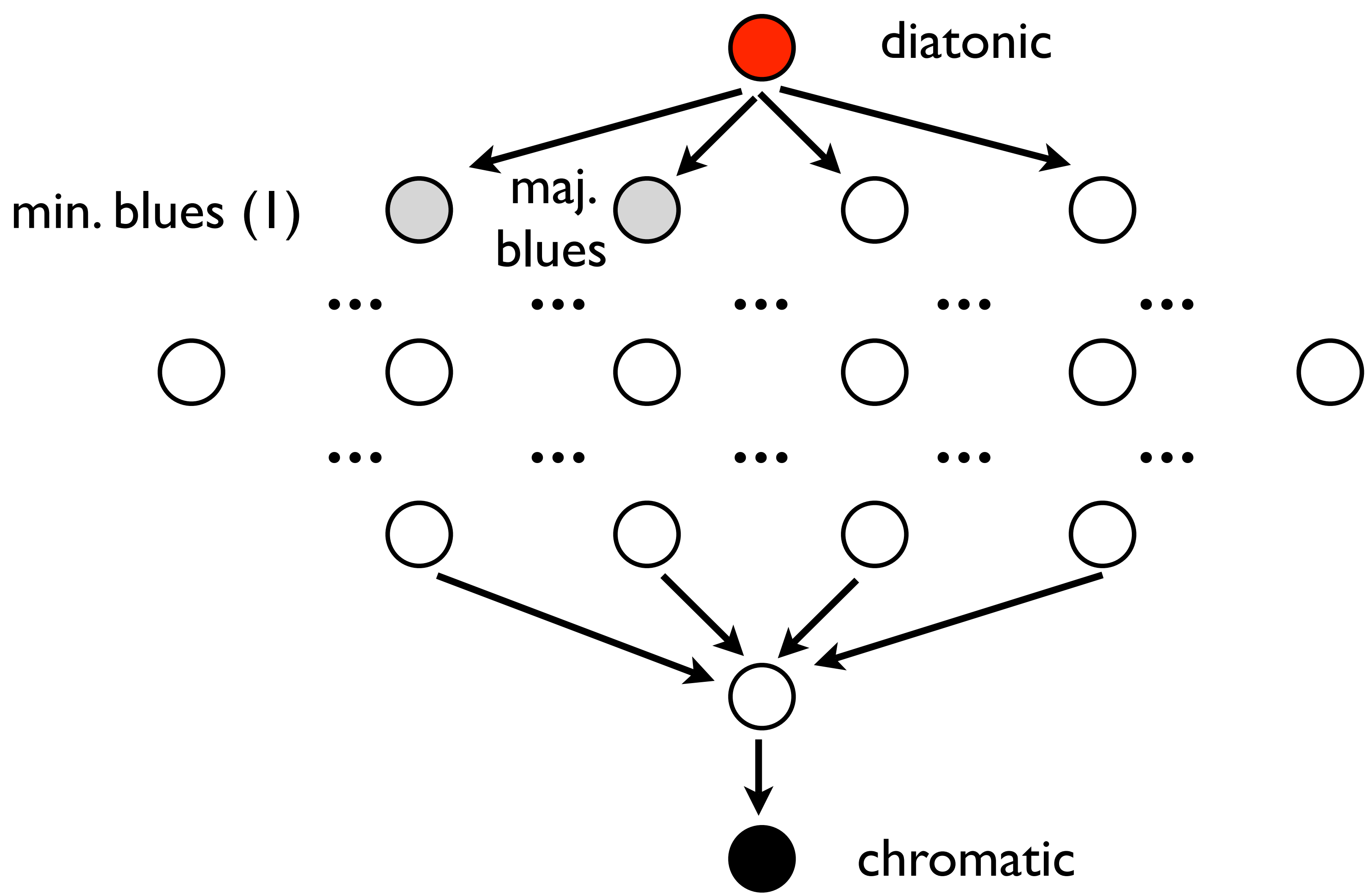
# A Partial Order on Scales

- Partial order from the whole tone scale



# A Partial Order on Scales

- Partial order from the diatonic scale



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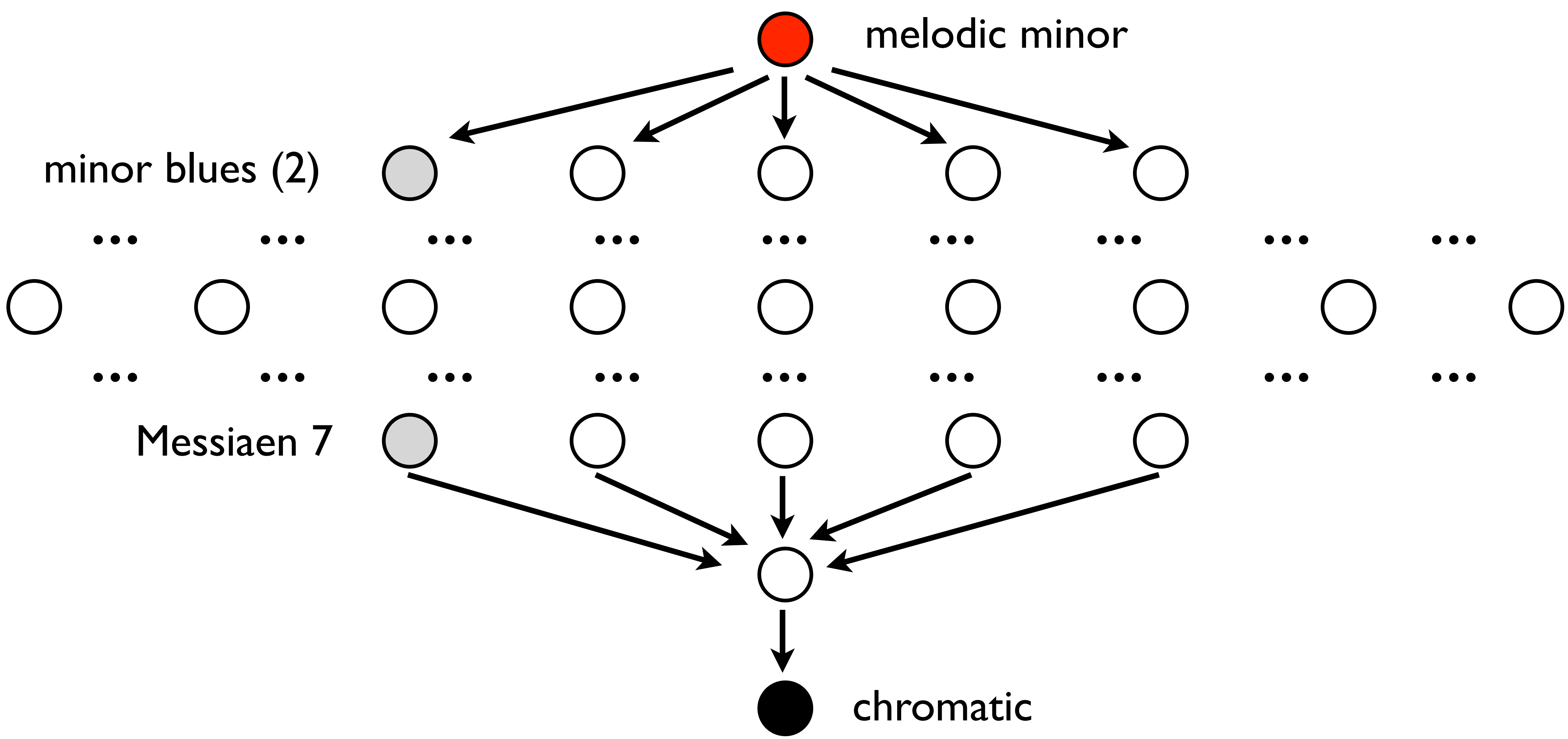
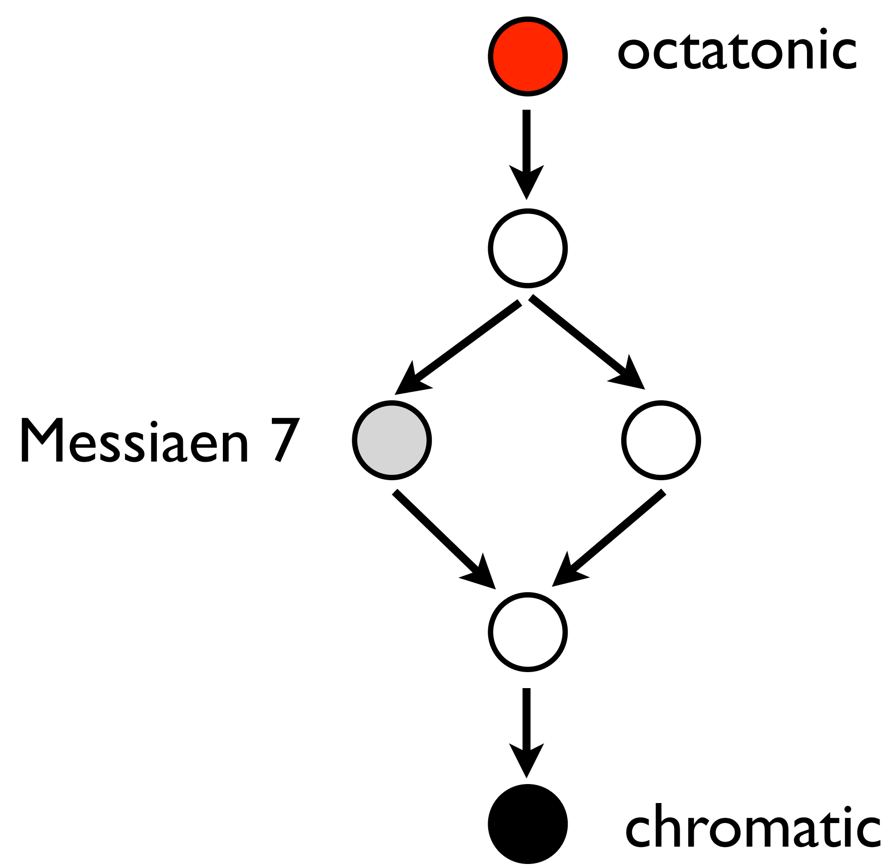
- Partial order from the melodic minor scale



# A Partial Order on Scales

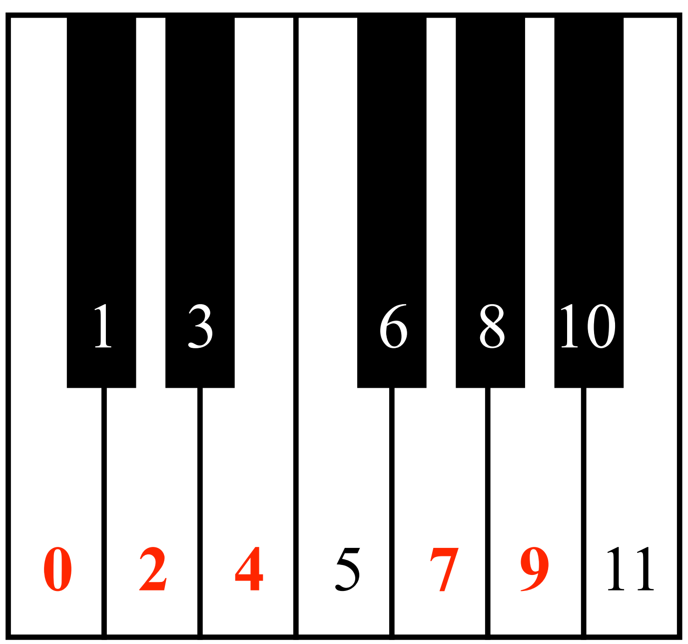
- Partial order from the octatonic scale



# A Partial Order on Scales

- Generalization: *Dense*($k$)*Scale*: a scale consisting solely of steps of size $1$, or $2$, or ... or $k$.

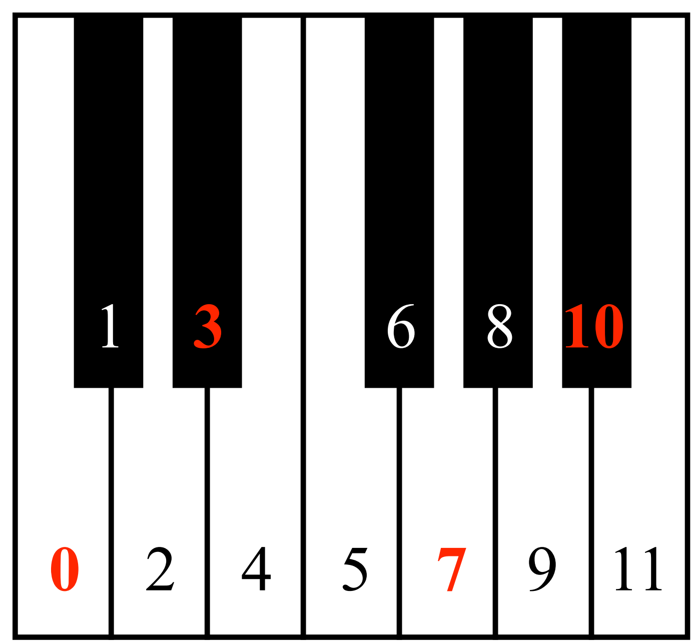
pentatonic




$(2, 2, 3, 2, 3)$

Dense(3)

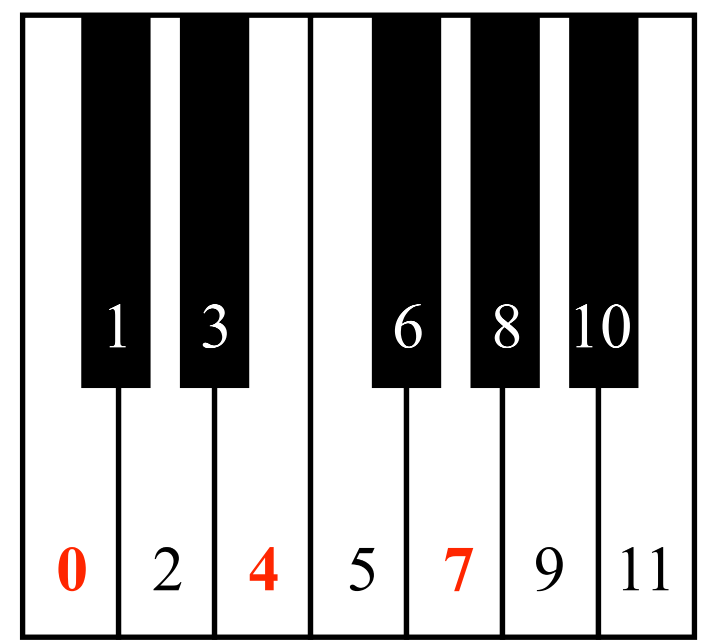
minor seventh chord




$(3, 4, 3, 2)$

Dense(4)

major triad




$(4, 3, 5)$

Dense(5)

# Partial Order modulo Group

Theorem 2

A scale in Dense($k$) mod $G$ is minimal if and only if every scalar third spans at least $k+1$ semitones.

# Partial Order modulo Group

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A scale in Dense($k$) mod $G$ is minimal if and only if every scalar third spans at least $k + 1$ semitones.

- A *scalar third* is the sum of two consecutive steps in a scale.
  - $(2,2,1,2,2,2,1)$ in Dense($2$) mod $G$ has scalar thirds $(4,3,3,4,4,3,3)$.

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  - $(2,2,1,2,2,2,1)$ in Dense($2$) mod $G$ has scalar thirds $(4,3,3,4,4,3,3)$.
- Theorem $2$ is true in $N$-tone equal temperament.

# A Partial Order on Scales

- Dense(3) has seven minimal elements:
  - (1,3,1,3,1,3), a symmetric scale (R. Daly, "Pulp Fiction")
  - (2,2,2,2,2,2), the whole tone scale
  - (2,2,3,2,3), the pentatonic scale
  - (2,2,3,3,2), the dominant ninth chord
  - (3,1,3,2,3), a blues scale
  - (3,1,3,3,2), the dominant seventh + sharp ninth chord (J. Hendrix, "Foxy Lady")
  - (3,3,3,3), the fully diminished chord

# A Partial Order on Scales

- Dense(4) has an additional six minimal elements:
  - (3,3,4,2), the minor seventh, flat fifth chord
  - (3,4,3,2), the minor seventh chord
  - (4,2,4,2), the dominant seventh, flat fifth chord
  - (4,3,3,2), the dominant seventh chord
  - (4,3,4,1), the major seventh chord
  - (4,4,4), the augmented triad

# A Partial Order on Scales

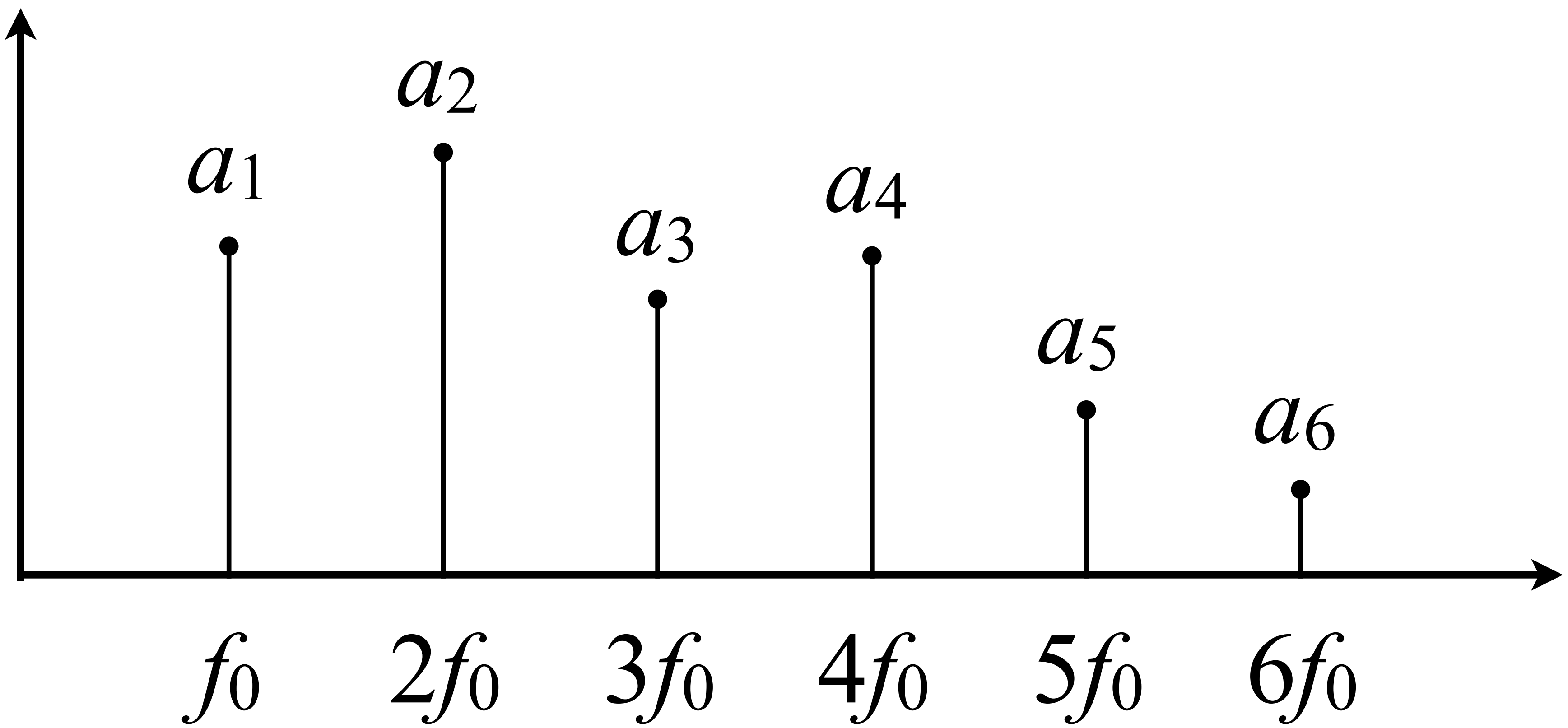
- Dense(5) has an additional four minimal elements:
  - (3,4,5), the minor triad
  - (4,3,5), the major triad
  - (5,1,5,1), a symmetric chord
  - (5,5,2), the quartal triad (H. Hancock, "Maiden Voyage")

# A Timbral Partial Order

- Timbre is the "characteristic sound" of a musical voice.
  - many aspects; notoriously difficult to quantify
- But musicians commonly speak about timbre in comparative ways
  - "a trumpet is brighter than a french horn"
  - "he sings like Bob Dylan with a head cold"
- Can we model these judgements using a partial order?

# A Timbral Partial Order

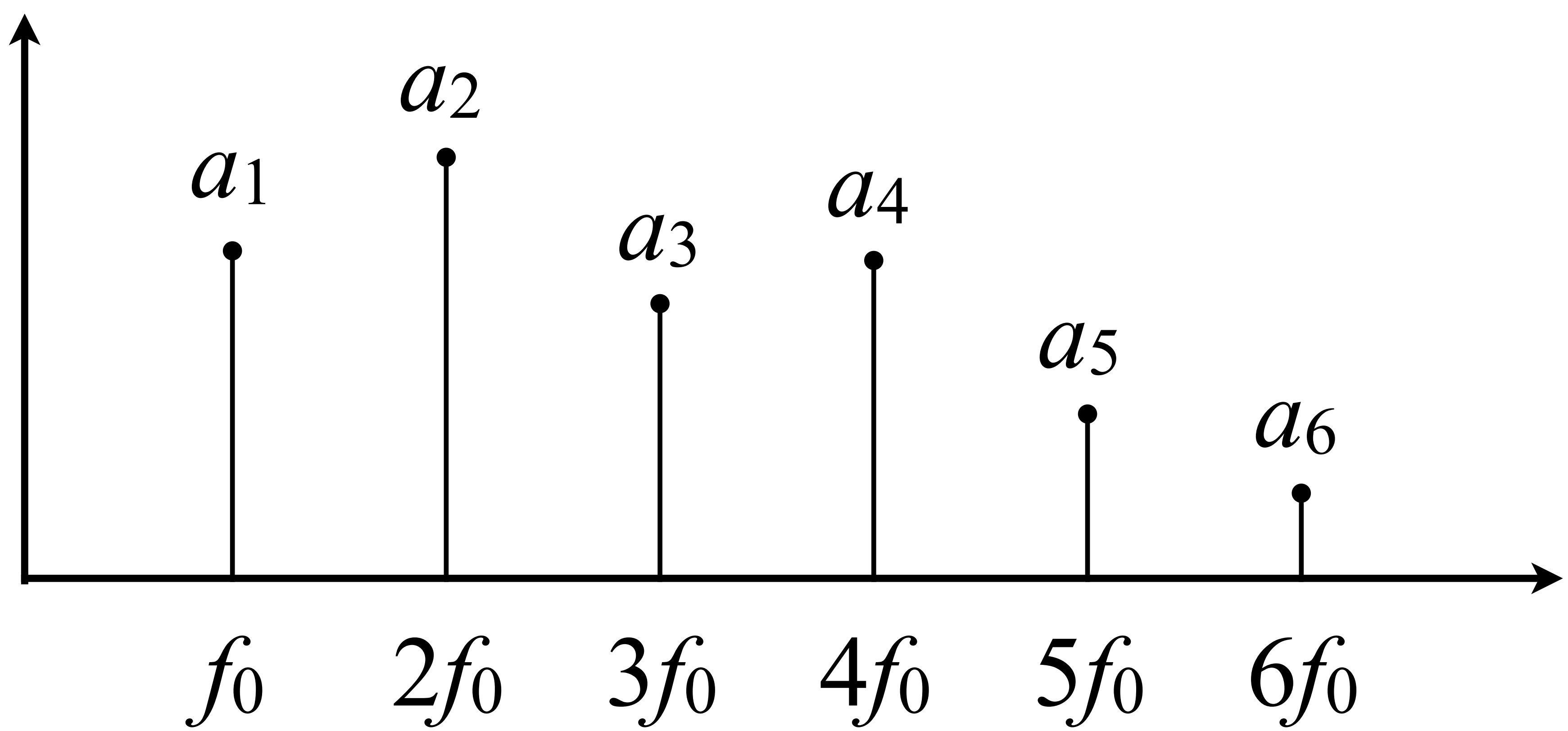
- Discrete power spectrum model for "steady-state timbre".




$a_k$ = power at $k^{\text{th}}$ harmonic

# A Timbral Partial Order

- Discrete power spectrum model for "steady-state timbre".




$\sum a_k = 1$ (unit volume)

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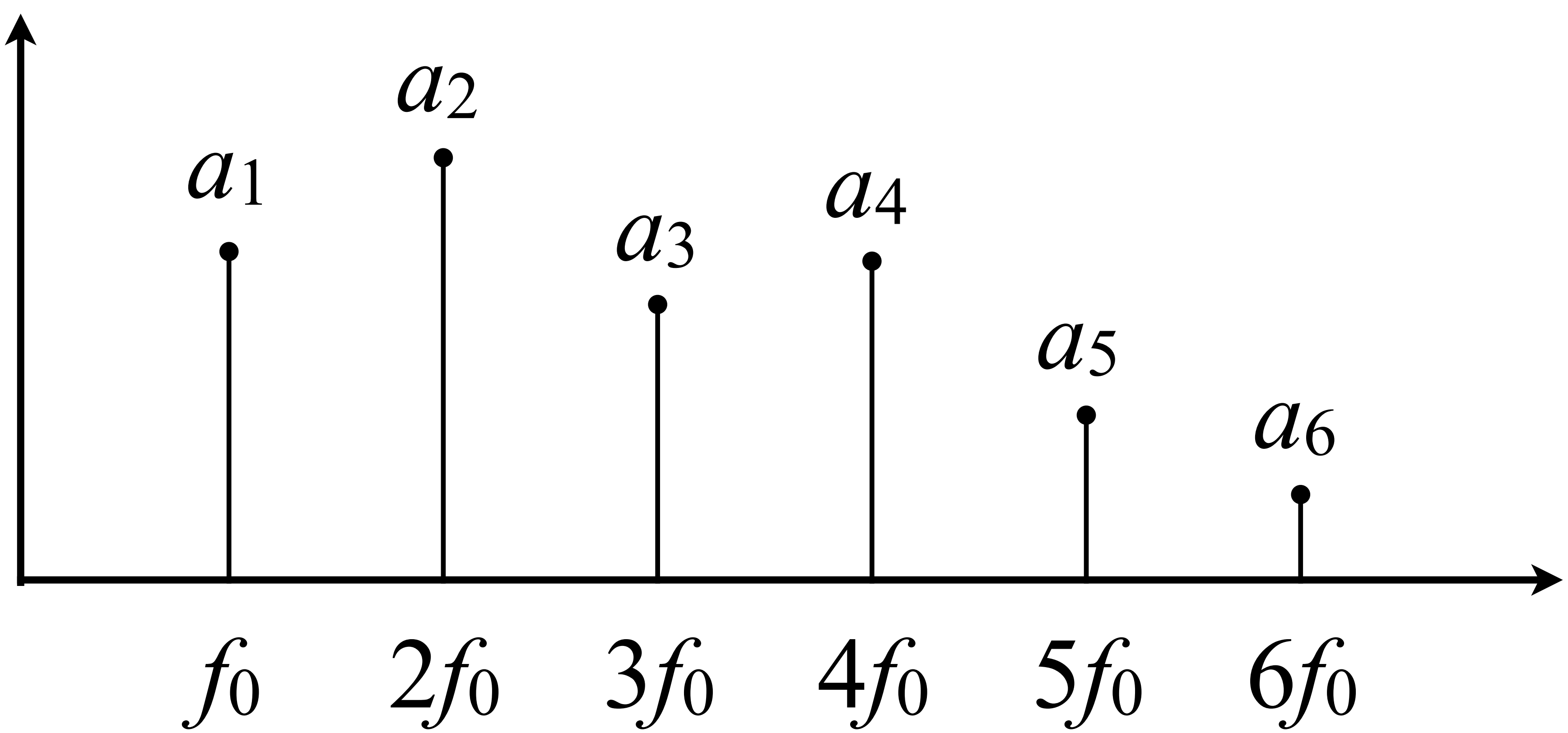


$a = (a_1, a_2, \ldots, a_n) =$ timbral vector

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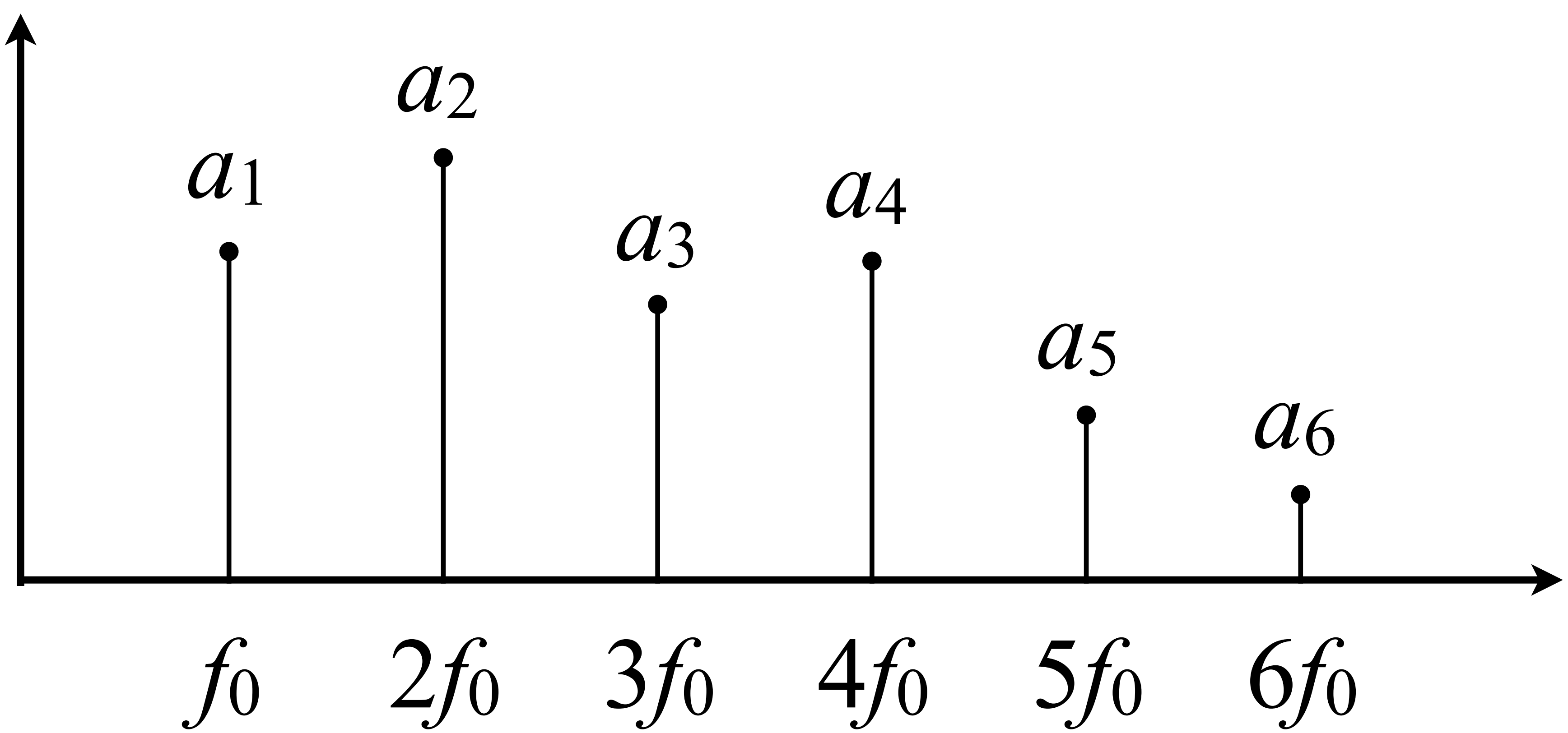
- Discrete power spectrum model for "steady-state timbre".




$$S = \{\text{all timbral vectors}\} = \{\text{all probability vectors}\}$$

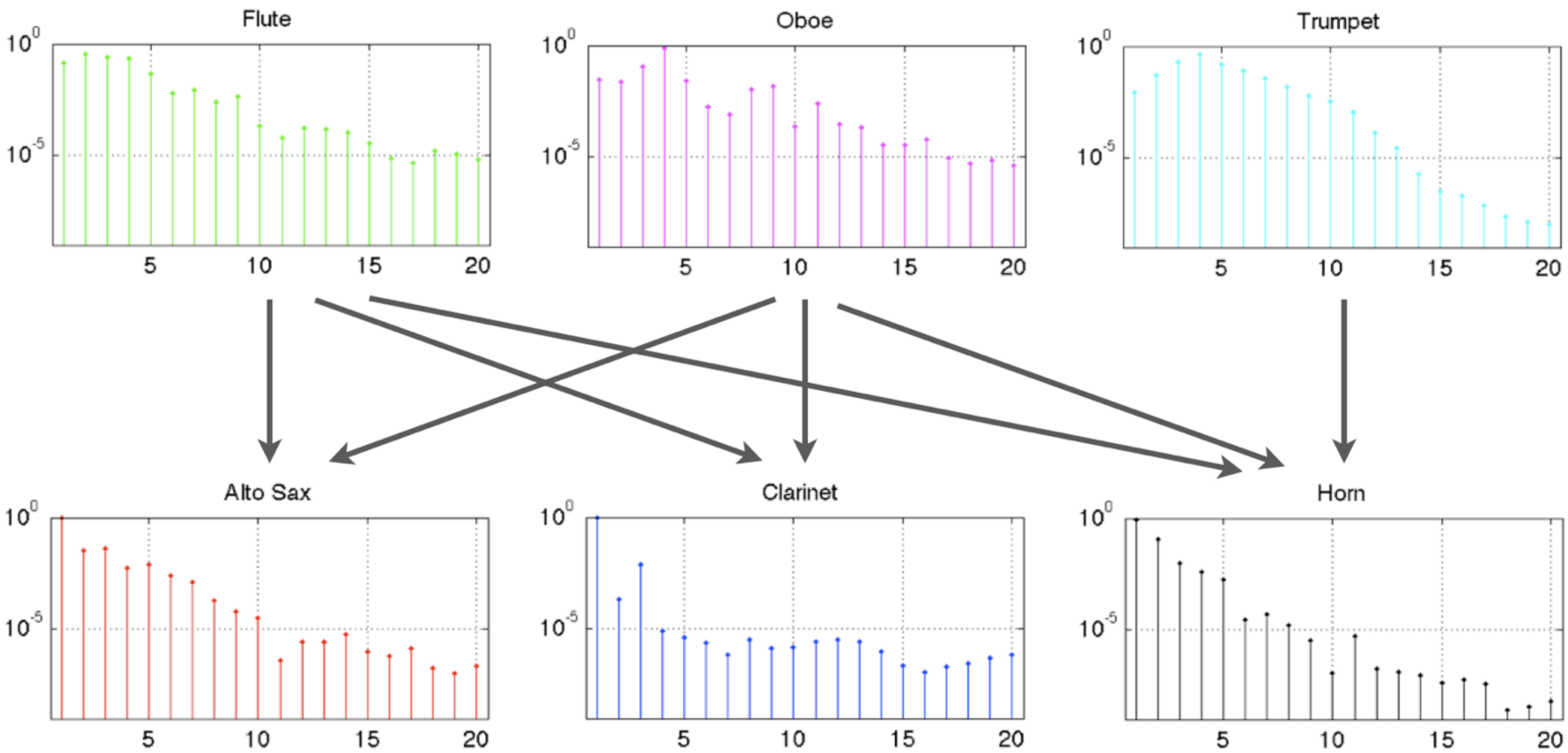
# A Timbral Partial Order

- "Brightness" aspect of timbre
  - refers to a prevalence of high harmonics in the sound
- The "Brightness" partial order
  - Timbral vector $b$ is "brighter than" timbral vector $a$ if

$$\sum_{j \geq k} a_j \leq \sum_{j \geq k} b_j \quad \forall k$$

  - I.e. every high-pass filter returns more power from $b$ than from $a$

# A Timbral Partial Order




Six Instruments in the "Brightness" Order

# Sound Design Problem

- *Among all instruments which are no brighter than a trumpet, which has the timbre that is closest to an oboe?*

- How do we measure "closeness"?

- Total Variational Distance

$$d_{\mathrm{TV}}(x, y) = \{\sum_{i \in I} |x_i - y_i| : I \subseteq 1, 2, \ldots, n\}$$

  - maximum power differential across subsets of harmonics

# Sound Design Problem

- Constrained Optimization Problem

Minimize: $d_{\mathrm{TV}}(x, \mathrm{oboe})$

Subject to: $x \leq$ trumpet in the "brightness" order

# Sound Design Problem

- Constrained Optimization Problem

$$\text{Minimize: } \|x - \text{oboe}\|_1$$
$$\text{Subject to: } Hx \leq H(\text{trumpet}) \text{ component-wise}$$

# Sound Design Problem

- Constrained Optimization Problem

$$\text{Minimize: } \|x - \text{oboe}\|_1$$
$$\text{Subject to: } Hx \leq H(\text{trumpet}) \text{ component-wise}$$

  - efficiently solvable via linear programming

# Sound Design Problem

Minimize: $\|x - \text{oboe}\|_1$

Subject to: $Hx \leq H(\text{trumpet})$ component-wise

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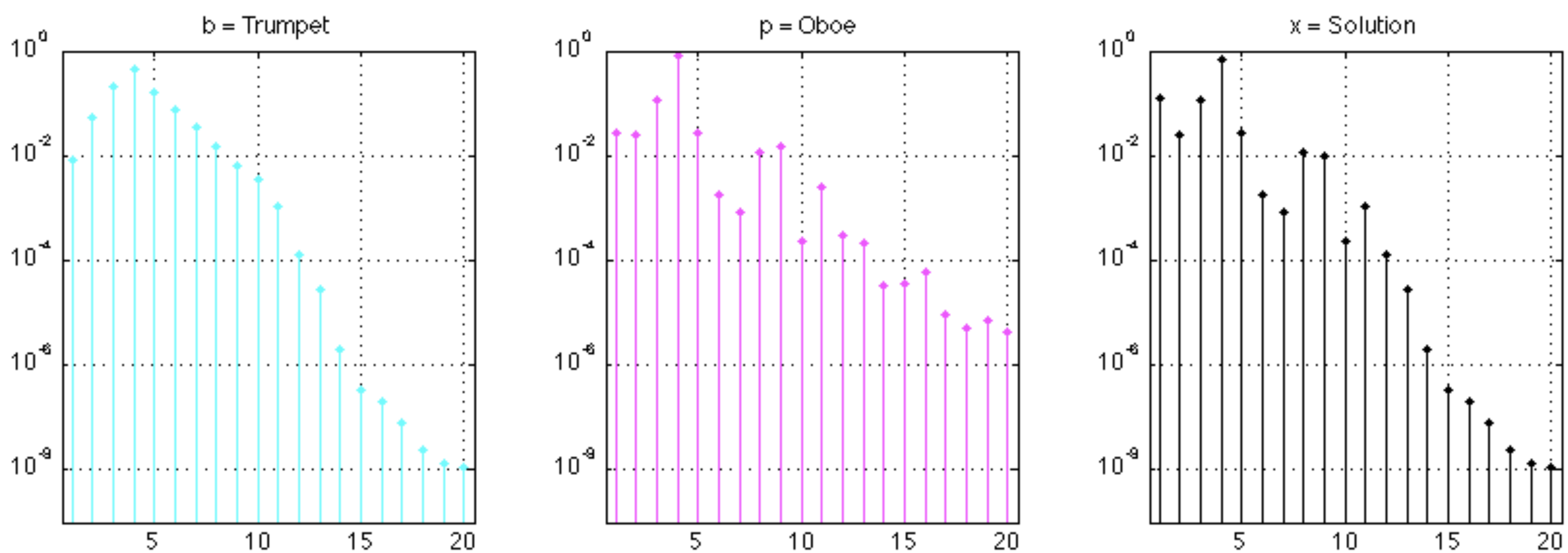
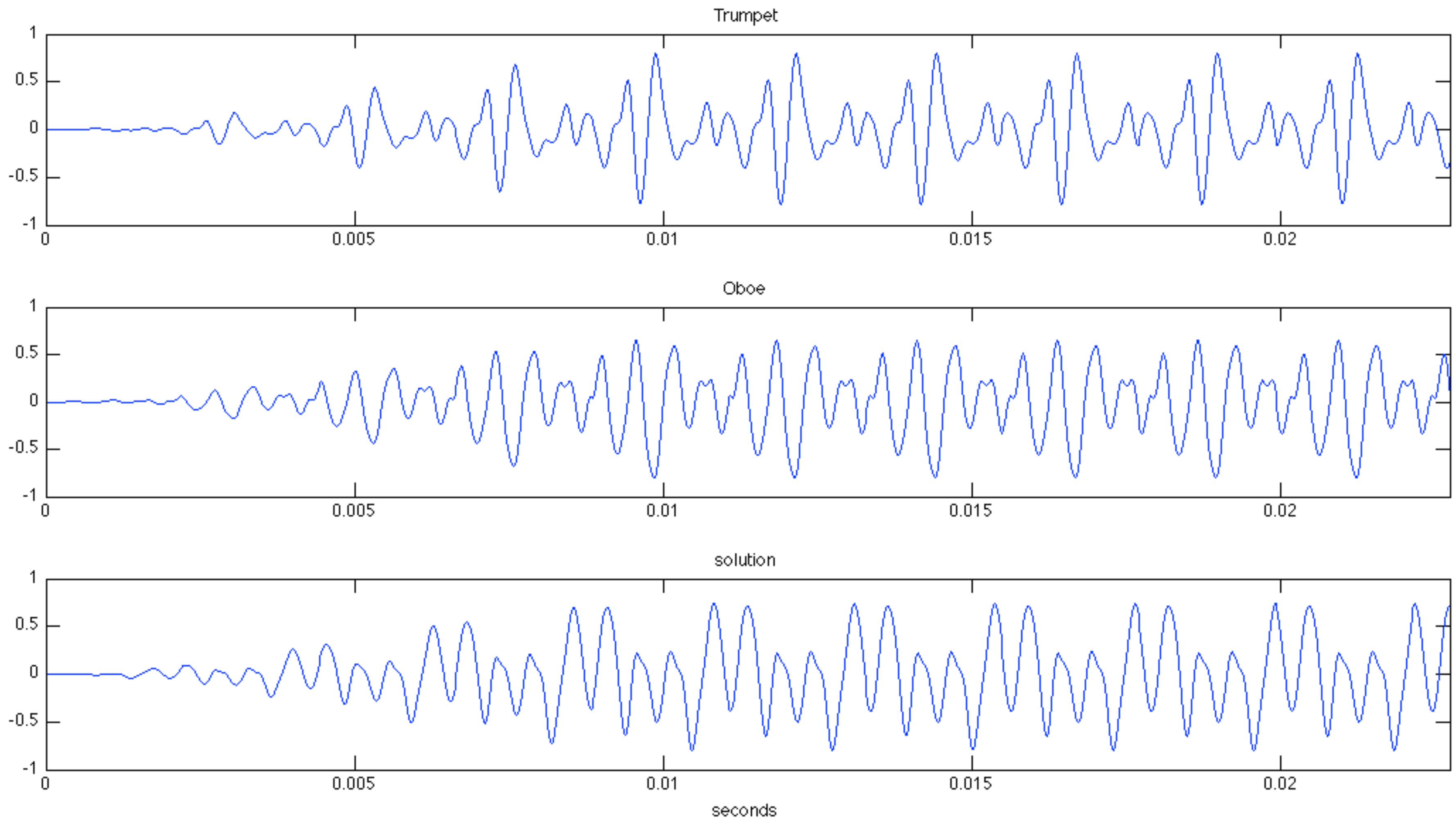



Solution to Sound Design Problem

# Sound Design Problem

Minimize: $\|x - \text{oboe}\|_1$

Subject to: $Hx \leq H(\text{trumpet})$ component-wise




Solution to Sound Design Problem

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Minimize: $\|x - \text{oboe}\|_1$

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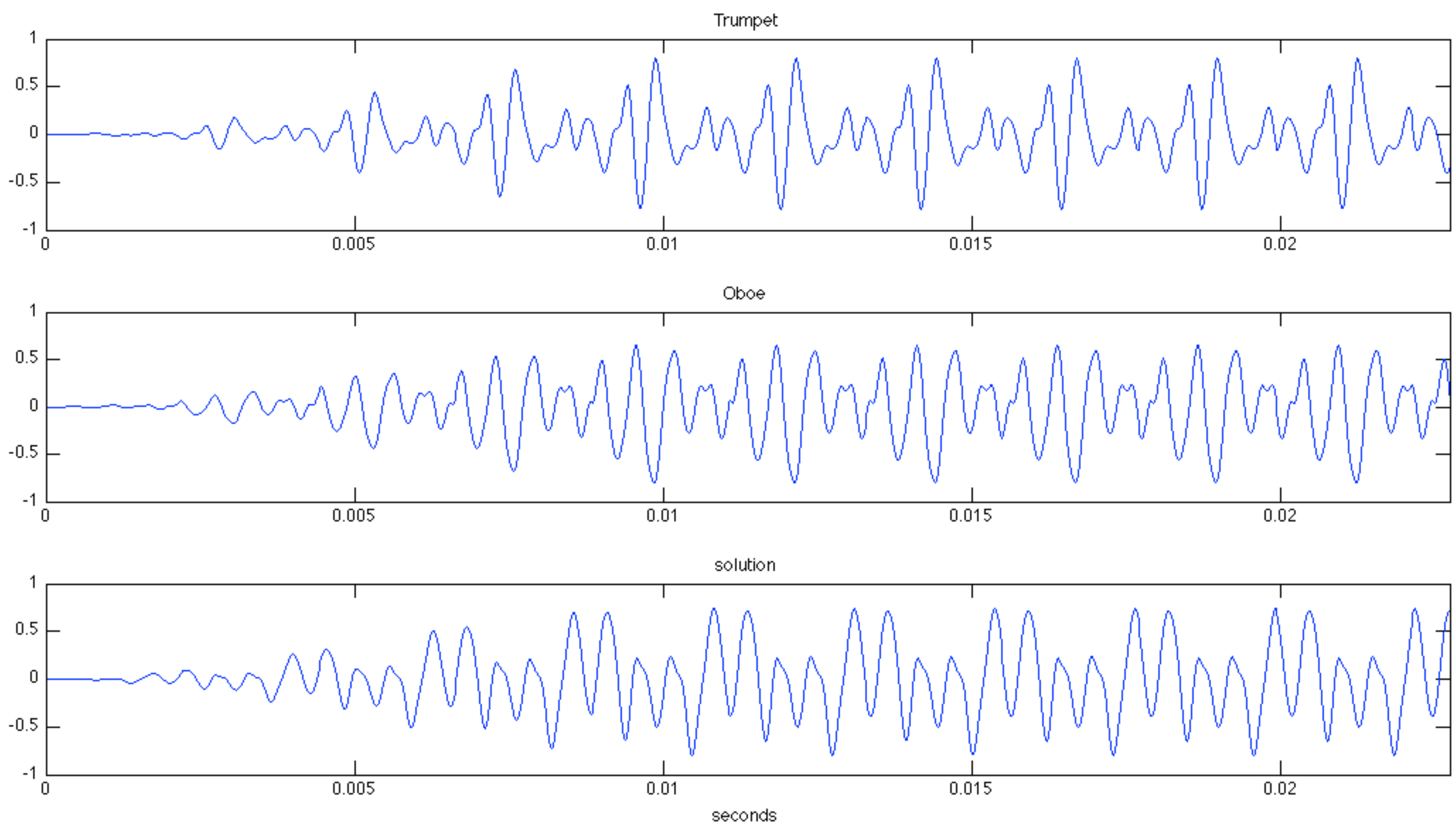


Solution to Sound Design Problem

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Minimize: $\|x - \text{oboe}\|_1$

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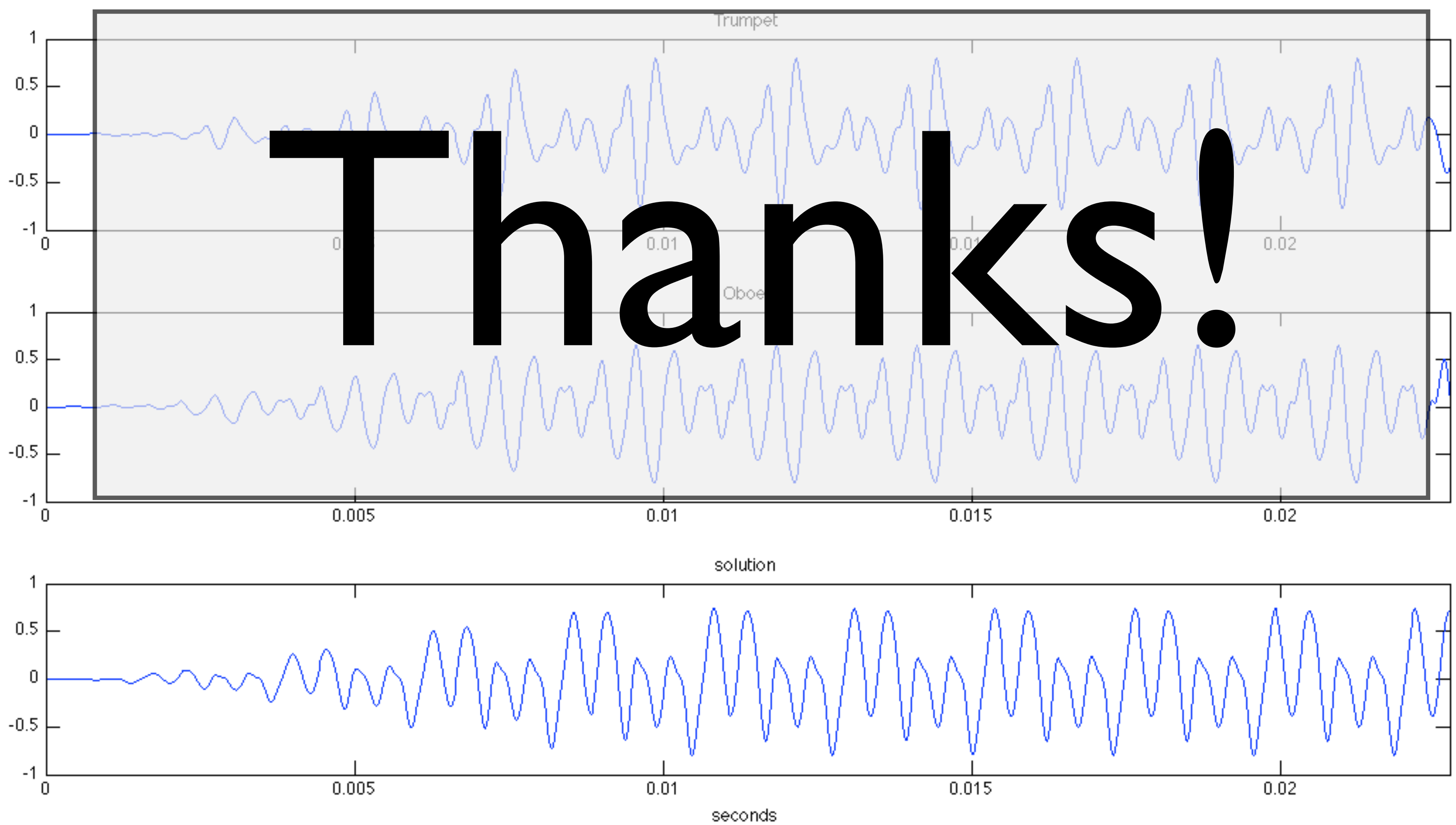


Solution to Sound Design Problem

# Some References


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