AN ELEMENTARY PROOF OF THE MEAN INEQUALITIES

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Abstract

In this paper our goal is to prove the well-known chain of inequalities involving the harmonic, geometric, logarithmic, identric, and arithmetic means using nothing more than basic calculus.

Abstract

Of course, these results are all well-known and several proofs of them and their generalizations have been given:

Hardy, Littlewood, and Polya (1964)

Carlson (1965)

Carlson and Tobey (1968)

Beckenbach and Bellman (1971)

Alzer (1985a, 1985b)

Abstract

Our purpose here is to present a unified approach and give the proofs as corollaries of one elementary theorem.

The Pythagorean Means

For a sequence of numbers $x = \{x_1, x_2, ..., x_n\}$ we will let

$$AM(x_1, x_2, ..., x_n) = AM(x) = \frac{\sum_{j-1}^{n} x_j}{n}$$

$$GM(x_1, x_2, ..., x_n) = GM(x) = \prod_{j=1}^{n} x_j^{1/n}$$

and

$$HM(x_1, x_2, ..., x_n) = HM(x) = \frac{n}{\sum_{j=1}^{n} \frac{1}{x_j}}$$

to denote the well-known *arithmetic*, *geometric*, and *harmonic means*, also called the *Pythagorean means* (PM).

Pythagorean Means

The Pythagorean means have the obvious properties:

1. $PM(x_1, x_2, \ldots, x_n)$ is independent of order

2. $PM(x, x, \ldots, x) = x$

3. $PM(bx_1, bx_2, \ldots, bx_n) = bPM(x_1, x_2, \ldots, x_n)$

Pythagorean Means

1.

2. $PM(x_1, x_2)$ is always a solution of a simple equation. In particular, the arithmetic mean of two numbers x_1 and x_2 can be defined via the equation

$$AM - x_1 = x_2 - AM$$

The harmonic mean satisfies the same relation with reciprocals, that is, it is a solution of the equation

$$\frac{1}{x_1} - \frac{1}{HM} = \frac{1}{HM} - \frac{1}{x_2}$$

The geometric mean of two numbers x_1 and x_2 can be visualized as the solution of the equation

$$\frac{x_1}{GM} = \frac{GM}{x_2}$$

Pythagorean Means

1. $GM = \sqrt{(AM)(HM)}$

2. $HM\left(x_1, \frac{1}{x_1}\right) = \frac{1}{AM(x_1, \frac{1}{x_1})}$

3. $(x_1 + x_2 + \cdots + x_n)\left(\frac{1}{x_1} + \frac{1}{x_2} + \cdots + \frac{1}{x_n}\right) \geq n^2$

This follows because

$$\frac{AM(x_1, x_2, \dots, x_n)}{HM(x_1, x_2, \dots, x_n)} \geq 1$$

Logarithmic and Identric Means

$$LM(0, x_2) = LM(x_1, 0) = 0$$

$$LM(x_1, x_1) = x_1$$

and for positive distinct numbers x_1 and x_2

$$LM(x_1, x_2) = \frac{x_2 - x_1}{ln x_2 - ln x_1}$$

Logarithmic and Identric Means

The following are some basic properties of the logarithmic means:

Logarithmic mean $LM(a, b)$ can be thought of as the mean-value of the function $f(x) = lnx$ over the interval $[a, b]$.

Logarithmic and Identric Means

The logarithmic mean can also be interpreted as the area under an exponential curve.

Since

$$\int_0^1 x^{1-t}\, y^t dt = x \int_0^1 \left(\frac{y}{x}\right)^t dt = \frac{x-y}{lnx - lny}$$

We also have the identity

$$LM(x,y) = \int_0^1 x^{1-t}\, y^t dt$$

Using this representation it is easy to show that

$$LM(cx, cy) = cLM(x,y)$$

Logarithmic and Identric Means

We have the identity

$$\frac{LM(x^2, y^2)}{LM(x,y)} = AM(x,y)$$

which follows easily:

$$\frac{LM(x^2, y^2)}{LM(x,y)} = \frac{x^2 - y^2}{2(lnx - lny)} \div \frac{x - y}{lnx - lny} = \frac{x + y}{2}$$

Logarithmic and Identric Means

The *identric mean* of two distinct positive real numbers x_1, x_2 is defined as:

$$IM(x_1, x_2) = \frac{1}{e}\left(\frac{{x_2}^{x_2}}{{x_1}^{x_1}}\right)^{\frac{1}{x_2 - x_1}}$$

with

$$IM(x_1, x_1) = x_1$$

Logarithmic and Identric Means

The slope of the secant line joining the points $(a, f(a))$ and $(b, f(b))$ on the graph of the function

$$f(x) = x \ln(x)$$

is the natural logarithm of $IM(a, b)$.

The Main Theorem

Theorem 1. *Suppose* $f : [a, b] \to \Re$ *is a function with a strictly increasing derivative. Then*

$$\int_a^b f(t)dt < \frac{b-a}{2}\left[f(s) - s\left(\frac{f(b)-f(a)}{b-a}\right) + \frac{bf(b)-af(a)}{b-a}\right]$$

for all $s < t$ *in* $[a, b]$.

The Main Theorem

Let s_0 be defined by the equation

$$f'(s_0) = \frac{f(b) - f(a)}{b - a}$$

Then,

$$\int_a^b f(t)dt < \frac{b-a}{2}\left[f(s_0) - s_0\left(\frac{f(b) - f(a)}{b - a}\right) + \frac{bf(b) - af(a)}{b - a}\right]$$

is the sharpest form of the above inequality.

The Main Theorem

Proof. By the Mean Value Theorem, for all s, t in $[a, b]$, we have

$$\frac{f(t) - f(s)}{t - s} < f'(u)$$

for some u between s and t. Assuming without loss of generality $s < t$, by the assumption of the theorem we have

$$f(t) - f(s) < (t - s)f'(t)$$

The Main Theorem

Integrating both sides with respect to t, we have

$$\int_a^b f(t)dt < (b-a)f(s) + bf(b) - af(a) - s[f(b) - f(a)] - \int_a^b f(t)dt$$

and the inequality of the theorem follows.

The Main Theorem

Let us now put

$$g(s) = (b-a)f(s) + bf(b) - af(a) - s[f(b) - f(a)] - \int_a^b f(t)dt$$

Note that

$$g'(s) = (b-a)f'(s) - [f(b) - f(a)]$$

The Main Theorem

Moreover, since

$$g(b) = g(a) = (b - a)[f(a) + f(b)] - \int_a^b f(t)dt$$

there exists an s_0 in (a, b) such that $g'(s_0) = 0$.

The Main Theorem

Since f' is strictly increasing, we have

$$g'(s) > g'(s_0) = 0$$

for $s > s_0$

and

$$g'(s) < g'(s_0) = 0$$

for $s < s_0$

Thus, s_0 is a minimum of g and $g(s_0) \leq g(s)$ for all s in $[a, b]$.

Proof of the Mean Inequalities

Let us now assume that $0 < a < b$.

Let us let $f(t) = \frac{1}{t^2}$. The condition of the Theorem 1 is satisfied.

We compute

$$s_0 = (abH)^{1/3}$$

Proof of the Mean Inequalities

And the inequality of the theorem becomes

$$HM(a, b) < GM(a, b)$$

Now, let us let $f(t) = \frac{1}{t}$. The condition of Theorem 1 is satisfied.

Proof of the Mean Inequalities

One can easily compute

$$s_0 = \sqrt{ab}$$

And the inequality of the theorem becomes

$$GM(a,b) < LM(a,b)$$

Proof of the Mean Inequalities

Now let $f(t) = -lnt$. Again the condition of Theorem 1 is satisfied. The s_0 of the theorem can be computed from the equation

$$s_0 = L$$

the logarithmic mean of a and b.

Proof of the Mean Inequalities

The inequality of the theorem becomes

$$LM(a,b) < IM(a,b)$$

Proof of the Mean Inequalities

Finally, let us put $f(t) = tlnt$. Again the condition of Theorem 1 is satisfied.

In this case,

$$s_0 = I$$

The identric mean of a and b.

Proof of the Mean Inequalities

The inequality of the theorem becomes

$$IM(a, b) < AM(a, b)$$

Proof of the Mean Inequalities

Thus, we now have for $a \neq b$

$$HM(a,b) < GM(a,b) < LM(a,b) < IM(a,b) < AM(a,b)$$

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